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M-Theory on the Orbifold $\mathbb{C}^2/\mathbb{Z}_N$

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Abstract

We construct M-theory on the orbifold $\mathbb{C}^2/\mathbb{Z}_N$ by coupling 11-dimensional supergravity to a seven-dimensional Yang-Mills theory located on the orbifold fixed plane. It is shown that the resulting action is supersymmetric to leading non-trivial order in the 11-dimensional Newton constant. This action provides the starting point for a reduction of M-theory on G_2 spaces with co-dimension four singularities.

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1 Introduction

It has been known for a long time that compactification of 11-dimensional supergravity on seven-dimensional manifolds leads to non-chiral four-dimensional effective theories [1, 2] and does, therefore, not provide a viable framework for particle phenomenology. With the construction of M-theory on the orbifold $S^1/\mathbb{Z}_2 \times \mathbb{R}^{1,9}$, Hořava and Witten [3] demonstrated for the first time that the situation can be quite different for compactifications on singular spaces. In fact, they showed that new states in the form of two 10-dimensional E_8 super-Yang-Mills multiplets, located on the two 10-dimensional fixed planes of this orbifold, had to be added to the theory for consistency and they explicitly constructed the corresponding supergravity theory by coupling 11-dimensional supergravity in the bulk to these super-Yang-Mills theories. It soon became clear that this theory allows for phenomenologically interesting Calabi-Yau compactifications [4]-[6] and, as the strong-coupling limit of the heterotic string, should be regarded as a promising avenue towards particle phenomenology from M-theory.

More recently, it has been discovered that phenomenologically interesting theories can also be obtained by M-theory compactification on singular spaces with G_2 holonomy [7]-[20]. More precisely, in such compactifications, certain co-dimension four singularities within the G_2 space lead to low-energy non-Abelian gauge fields [7, 8] while co-dimension seven singularities can lead to matter fields [9, 10, 8]. In contrast, M-theory compactifications on G_2 manifolds reduce to four-dimensional $\mathcal{N} = 1$ theories with only Abelian vector multiplets and uncharged chiral multiplets. Our focus in the present paper will be the non-Abelian gauge fields arising from co-dimension four singularities. The structure of the G_2 space close to such a singularity is of the form $\mathbb{C}^2/D \times B$, where D is one of the discrete ADE subgroups of SU(2) and B is a three-dimensional space. We will, for simplicity, focus on A-type singularities, that is $D = \mathbb{Z}_N$, which lead to gauge fields with gauge group SU(N). A large class of singular G_2 spaces containing such singularities has been obtained in [21], by orbifolding seven tori [22]. It was shown that, within this class of examples, the possible values of N are 2, 3, 4 and 6. However, in this paper, we keep N general, given that there may be other constructions which lead to more general N. The gauge fields are located at the fixed point of $\mathbb{C}^2/\mathbb{Z}_N$ (the origin of $\mathbb{C}^2$) times $B \times \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the four-dimensional uncompactified space-time, and are, hence, seven-dimensional in nature. One would, therefore, expect there to exist a supersymmetric theory which couples M-theory on the orbifold $\mathbb{C}^2/\mathbb{Z}_N$ to seven-dimensional super-Yang-Mills theory. It is the main purpose of the present paper to construct this theory explicitly.

Although motivated by the prospect of applications to G_2 compactifications, we will formulate this problem in a slightly more general context, seeking to understand the general structure of low-energy M-theory on orbifolds of ADE type. Concretely, we will construct 11-dimensional supergravity on the orbifold $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{R}^{1,6}$ coupled to seven-dimensional SU(N) super-Yang-Mills theory located on the orbifold fixed plane $\{\mathbf{0}\} \times \mathbb{R}^{\mathbf{1,6}}$. For ease of terminology, we will also refer to this orbifold plane, somewhat loosely as the "brane". This result can then be applied to compacitifications of M-theory on G_2 spaces with $\mathbb{C}^2/\mathbb{Z}_N$ singularities, as well as to other problems (for example M-theory on certain singular limits of K3). We stress that this construction is very much in the spirit of the Hořava-Witten theory [3], which couples 11-dimensional supergravity on $S^1/\mathbb{Z}_2 \times \mathbb{R}^{1,9}$ to 10-dimensional super-Yang-Mills theory.

Let us briefly outline our method to construct this theory which relies on combining information from the known actions of 11-dimensional [23, 24] and seven-dimensional supergravity [25]-[27]. Firstly, we constrain the field content of 11-dimensional supergravity (the "bulk fields") to be com-

patible with the $\mathbb{Z}_N$ orbifolding symmetry. We will call the Lagrangian for this constrained version of 11-dimensional supergravity $\mathcal{L}_{11}$. As a second step this action is truncated to seven dimensions, by requiring all fields to be independent of the coordinates y of the orbifold $\mathbb{C}^2/\mathbb{Z}_N$ (or, equivalently, constraining it to the orbifold plane at $y = 0$). The resulting Lagrangian, which we call $\mathcal{L}_{11}|_{y=0}$, is invariant under half of the original 32 supersymmetries and represents a seven-dimensional $\mathcal{N} = 1$ supergravity theory which turns out to be coupled to a single $U(1)$ vector multiplet for $N > 2$ or three $U(1)$ vector multiplets for $N = 2$. As a useful by-product, we obtain an explicit identification of the (truncated) 11-dimensional bulk fields with the standard fields of 7-dimensional Einstein-Yang-Mills (EYM) supergravity. We know that the additional states on the orbifold fixed plane should form a seven-dimensional vector multiplet with gauge group SU(N). In a third step, we couple these additional states to the truncated seven-dimensional bulk theory $\mathcal{L}_{11}|_{y=0}$ to obtain a seven-dimensional EYM supergravity $\mathcal{L}_{SU(N)}$ with gauge group $U(1) \times SU(N)$ for $N > 2$ or $U(1)^3 \times SU(N)$ for $N = 2$. We note that, given a fixed gauge group the structure of $\mathcal{L}_{SU(N)}$ is essentially determined by seven-dimensional supergravity. We further write this theory in a form which explicitly separates the bulk degrees of freedom (which we have identified with 11-dimensional fields) from the degrees of freedom in the $SU(N)$ vector multiplets. Given this preparation we prove in general that the action

$$S_{11-7} = \int_{\mathcal{M}_{11}} \mathrm{d}^{11}x \left[\mathcal{L}_{11} + \delta^{(4)}(y^A) \left(\mathcal{L}_{\mathrm{SU(N)}} - \kappa^{8/9}\mathcal{L}_{11}\right)\right] \tag{1.1}$$

is supersymmetric to leading non-trivial order in an expansion in κ, the 11-dimensional Newton constant. Inserting the various Lagrangians with the appropriate field identifications into this expression then provides us with the final result.

The plan of the paper is as follows. In Section 2 we remind the reader of the action of 11-dimensional supergravity. As mentioned above this is to be our bulk theory. We then go on to discuss the constraints that arise on the fields from putting this theory on the orbifold. We also lay out our conventions for rewriting 11-dimensional fields according to a seven plus four split of the coordinates. In Section 3 we examine our bulk Lagrangian constrained to the orbifold plane and recast it in standard seven-dimensional form. The proof that the action (1.1) is indeed supersymmetric to leading non-trivial order is presented in Section 4. Finally, in Section 5 we present the explicit result for the coupled 11-/7-dimensional action and the associated supersymmetry transformations. We end with a discussion of our results and an outlook on future directions. Three appendices present technical background material. In Appendix A we detail our conventions for spinors in eleven, seven and four dimensions and describe how we decompose 11-dimensional spinors. We also give some useful spinor identities. In Appendix B we have collected some useful group-theoretical information related to the cosets $SO(3, M)/SO(3) \times SO(M)$ of $d = 7$ EYM supergravity which will be used in the reduction of the bulk theory to seven-dimensions. The final Appendix is a self-contained introduction to EYM supergravity in seven dimensions.

2 Eleven-dimensional supergravity on the orbifold

In this section we begin our discussion of M-theory on $\mathcal{M}_{11}^N = \mathbb{R}^{1,6} \times \mathbb{C}^2/\mathbb{Z}_N$ by describing the bulk action and the associated bulk supersymmetry transformations. We recall that fields propagating on orbifolds are subject to certain constraints on their configurations and proceed by listing and explaining these. First however we lay out our conventions, and briefly describe the decomposition

of spinors in a four plus seven split of the coordinates.

We take space-time to have mostly positive signature, that is $(-++\ldots+)$, and use indices $M,N,\ldots=0,1,\ldots,10$ to label the 11-dimensional coordinates (x^M). It is often convenient to split these into four coordinates y^A, where $A,B,\ldots=7,8,9,10$, in the directions of the orbifold $\mathbb{C}^2/\mathbb{Z}_N$ and seven remaining coordinates x^μ, where $\mu,\nu,\ldots=0,1,2,\ldots,6$, on $\mathbb{R}^{1,6}$. Frequently, we will also use complex coordinates $(z^p,\bar{z}^{\bar{p}})$ on $\mathbb{C}^2/\mathbb{Z}_N$, where $p,q,\ldots=1,2$, and $\bar{p},\bar{q},\ldots=\bar{1},\bar{2}$ label holomorphic and anti-holomorphic coordinates, respectively. Underlined versions of all the above index types denote the associated tangent space indices.

All 11-dimensional spinors in this paper are Majorana. Having split coordinates into four- and seven-dimensional parts it is useful to decompose 11-dimensional Majorana spinors accordingly as tensor products of $SO(1,6)$ and $SO(4)$ spinors. To this end, we introduce a basis of left-handed $SO(4)$ spinors $\{\rho^i\}$ and their right-handed counterparts $\{\rho^{\bar{\jmath}}\}$ with indices $i,j,\ldots=1,2$ and $\bar{\imath},\bar{\jmath},\ldots=\bar{1},\bar{2}$. Up to an overall rescaling, this basis can be defined by the relations $\gamma^{\underline{A}}\rho^i=\left(\gamma^{\underline{A}}\right)_{\bar{\jmath}}{}^{i}\rho^{\bar{\jmath}}$. An 11-dimensional spinor ψ can then be written as

$$\psi=\psi_i(x,y)\otimes\rho^i+\psi_{\bar{\jmath}}(x,y)\otimes\rho^{\bar{\jmath}}, \tag{2.1}$$

where the 11-dimensional Majorana condition on ψ forces $\psi_i(x,y)$ and $\psi_{\bar{\jmath}}(x,y)$ to be $SO(1,6)$ symplectic Majorana spinors. In the following, for any 11-dimensional Majorana spinor we will denote its associated seven-dimensional symplectic Majorana spinors by the same symbol with additional i and $\bar{\imath}$ indices. A full account of spinor conventions used in this paper, together with a derivation of the above decomposition can be found in Appendix A.

With our conventions in place, we proceed by reviewing 11-dimensional supergravity [23, 24, 3]. Its field content consists of the vielbein $e_M{}^{\underline{M}}$ and associated metric $g_{MN}=\eta_{\underline{MN}}e_M{}^{\underline{M}}e_N{}^{\underline{N}}$, the three-form field C, with field strength $G=\mathrm{d}C$, and the gravitino Ψ_M. In this paper we shall not compute any four fermi terms (and associated cubic fermion terms in the supersymmetry transformations), and so we shall neglect them throughout. The action is given by

$$\begin{aligned}\mathcal{S}_{11} &= \frac{1}{\kappa^2}\int_{\mathcal{M}_{11}^N}\mathrm{d}^{11}x\sqrt{-g}\biggl(\frac{1}{2}R-\frac{1}{2}\bar{\Psi}_M\Gamma^{MNP}\nabla_N\Psi_P-\frac{1}{96}G_{MNPQ}G^{MNPQ}\\ &\qquad -\frac{1}{192}\Bigl(\bar{\Psi}_M\Gamma^{MNPQRS}\Psi_S+12\bar{\Psi}^N\Gamma^{PQ}\Psi^R\Bigr)G_{NPQR}\biggr)\\ &\quad -\frac{1}{12\kappa^2}\int_{\mathcal{M}_{11}^N}C\wedge G\wedge G+\ldots\end{aligned} \tag{2.2}$$

where the dots stand for terms quartic in the gravitino. Here κ is the 11-dimensional Newton constant, $\Gamma^{M_1M_2\ldots M_n}$ denote anti-symmetrized products of gamma matrices in the usual way, and ∇_I is the spinor covariant derivative, defined in terms of the spin connection ω by

$$\nabla_M=\partial_M+\frac{1}{4}\omega_M{}^{\underline{MN}}\Gamma_{\underline{MN}}. \tag{2.3}$$

The transformation laws of local supersymmetry, parameterized by the spinor η, read

$$\begin{aligned}
\delta e_M{}^{\underline{N}} &= \bar{\eta}\Gamma^{\underline{N}}\Psi_M \\
\delta C_{MNP} &= -3\bar{\eta}\Gamma_{[MN}\Psi_{P]} \\
\delta\Psi_M &= 2\nabla_M\eta + \frac{1}{144}\left(\Gamma_M{}^{NPQR} - 8\delta_M^N\Gamma^{PQR}\right)\eta G_{NPQR} + \dots,
\end{aligned} \tag{2.4}$$

where the dots denote terms cubic in the gravitino.

In order for the above bulk theory to be consistent on the orbifold $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{R}^{1,6}$ we need to constrain fields in accordance with the $\mathbb{Z}_N$ orbifold action. Let us now discuss in detail how this works.

We denote by R the $SO(4)$ matrix of order N that generates the $\mathbb{Z}_N$ symmetry on our orbifold. This matrix acts on the 11-dimensional coordinates as $(x, y) \to (x, Ry)$ which implies the existence of a seven-dimensional fixed plane characterized by $\{y = 0\}$. For a field X to be well-defined on the orbifold it must satisfy

$$X(x, Ry) = \Theta(R)X(x, y) \tag{2.5}$$

for some linear operator $\Theta(R)$ that represents the generator of $\mathbb{Z}_N$. In constructing our theory we have to choose, for each field, a representation Θ of $\mathbb{Z}_N$ for which we wish to impose this constraint. For the theory to be well-defined, these choices of representations must be such that the action (2.2) is invariant under the $\mathbb{Z}_N$ orbifold symmetry. Concretely, what we do is choose how each index type transforms under $\mathbb{Z}_N$. We take $R \equiv (R^A{}_B)$ to be the transformation matrix acting on curved four-dimensional indices $A, B, \dots$ while the generator acting on tangent space indices $\underline{A}, \underline{B}, \dots$ is some other $SO(4)$ matrix $T^{\underline{A}}{}_{\underline{B}}$. It turns out that this matrix must be of order N for the four-dimensional components of the vielbein to remain non-singular at the orbifold fixed plane. Seven-dimensional indices $\mu, \nu, \dots$ transform trivially. Following the correspondence Eq. (2.1), 11-dimensional Majorana spinors ψ are described by two pairs ψ_i and $\psi_{\bar{\imath}}$ of seven-dimensional symplectic Majorana spinor. We should, therefore, specify how the $\mathbb{Z}_N$ symmetry acts on indices of type i and $\bar{\imath}$. Supersymmetry requires that at least some spinorial degrees of freedom survive at the orbifold fixed plane. For this to be the case, one of these type of indices, i say, must transform trivially. Invariance of fermionic terms in the action (2.2) requires that the other indices, that is those of type $\bar{\imath}$, be acted upon by a $U(2)$ matrix $S_{\bar{\imath}}{}^{\bar{\jmath}}$ that satisfies the equation

$$S_{\bar{\imath}}{}^{\bar{k}}\left(\gamma^{\underline{A}}\right)_{\bar{k}}{}^{\jmath} = T^{\underline{A}}{}_{\underline{B}}\left(\gamma^{\underline{B}}\right)_{\bar{\imath}}{}^{\jmath}. \tag{2.6}$$

Given this basic structure, the constraints satisfied by the fields are as follows

$$e_\mu{}^{\underline{\nu}}(x, Ry) = e_\mu{}^{\underline{\nu}}(x, y), \tag{2.7}$$

$$e_A{}^{\underline{\nu}}(x, Ry) = (R^{-1})_A{}^B e_B{}^{\underline{\nu}}(x, y), \tag{2.8}$$

$$e_\mu{}^{\underline{A}}(x, Ry) = T^{\underline{A}}{}_{\underline{B}} e_\mu{}^{\underline{B}}(x, y), \tag{2.9}$$

$$e_A{}^{\underline{B}}(x, Ry) = (R^{-1})_A{}^C T^{\underline{B}}{}_{\underline{D}} e_C{}^{\underline{D}}(x, y), \tag{2.10}$$

$$
\begin{aligned}
C_{\mu\nu\rho}(x,Ry) &= C_{\mu\nu\rho}(x,y), &(2.11)\\
C_{\mu\nu A}(x,Ry) &= (R^{-1})_A{}^B C_{\mu\nu B}(x,y), \quad \text{etc.} &(2.12)\\
\Psi_{\mu i}(x,Ry) &= \Psi_{\mu i}(x,y), &(2.13)\\
\Psi_{\mu\bar{\imath}}(x,Ry) &= S_{\bar{\imath}}{}^{\bar{\jmath}}\Psi_{\mu\bar{\jmath}}(x,y), &(2.14)\\
\Psi_{Ai}(x,Ry) &= (R^{-1})_A{}^B\Psi_{Bi}(x,y), &(2.15)\\
\Psi_{A\bar{\imath}}(x,Ry) &= (R^{-1})_A{}^B S_{\bar{\imath}}{}^{\bar{\jmath}}\Psi_{B\bar{\jmath}}(x,y). &(2.16)
\end{aligned}
$$

Furthermore, covariance of the supersymmetry transformation laws with respect to $\mathbb{Z}_N$ requires

$$
\begin{aligned}
\eta_i(x,Ry) &= \eta_i(x,y), &(2.17)\\
\eta_{\bar{\imath}}(x,Ry) &= S_{\bar{\imath}}{}^{\bar{\jmath}}\eta_{\bar{\jmath}}(x,y). &(2.18)
\end{aligned}
$$

In complex coordinates $(z^p, \bar{z}^{\bar{p}})$, it is convenient to represent R by the following matrices

$$
(R^p{}_q) = e^{2i\pi/N}\mathbf{1}_2, \quad (R^{\bar{p}}{}_{\bar{q}}) = e^{-2i\pi/N}\mathbf{1}_2, \quad (R^{\bar{p}}{}_q) = (R^p{}_{\bar{q}}) = 0\,. \qquad (2.19)
$$

Using this representation, the constraint (2.10) implies

$$
e_p{}^{\underline{A}} = e^{-2i\pi/N} T^{\underline{A}}{}_{\underline{B}}\, e_p{}^{\underline{B}}\,. \qquad (2.20)
$$

Hence, for the vierbein $e_{\underline{A}}{}^{\underline{B}}$ to be non-singular T must have two eigenvalues $e^{2i\pi/N}$. Similarly, the conjugate of the above equation shows that T should have two eigenvalues $e^{-2i\pi/N}$. Therefore, in an appropriate basis we can use the following representation

$$
(T^{\underline{p}}{}_{\underline{q}}) = e^{2i\pi/N}\mathbf{1}_2\,, \qquad (T^{\underline{\bar{p}}}{}_{\underline{\bar{q}}}) = e^{-2i\pi/N}\mathbf{1}_2\,. \qquad (2.21)
$$

Given these representations for R and T, the matrix S is uniquely fixed by Eq. (2.6) to be

$$
(S_{\underline{i}}{}^{\underline{j}}) = e^{2i\pi/N}\mathbf{1}_2\,. \qquad (2.22)
$$

We will use the explicit form of R, T and S above to analyze the degrees of freedom when we truncate fields to be y independent.

When the 11-dimensional fields are taken to be independent of the orbifold y coordinates, the constraints (2.7)–(2.16) turn into projection conditions, which force certain field components to vanish. As we will see shortly, the surviving field components fit into seven-dimensional $\mathcal{N} = 1$ supermultiplets, a confirmation that we have chosen the orbifold $\mathbb{Z}_N$ action on fields compatible with supersymmetry. More precisely, for the case $N > 2$, we will find a seven-dimensional gravity multiplet and a single $U(1)$ vector multiplet. Hence, we expect the associated seven-dimensional $\mathcal{N} = 1$ Einstein-Yang-Mills (EYM) supergravity to have gauge group $U(1)$. For $\mathbb{Z}_2$ the situation is slightly more complicated, since, unlike for $N > 2$, some of the field components which transform bi-linearly under the generators are now invariant. This leads to two additional vector multiplets, so that the associated theory is a seven-dimensional $\mathcal{N} = 1$ EYM supergravity with gauge group $U(1)^3$. In the following section, we will write down this seven-dimensional theory, both for $N = 2$ and $N > 2$, and find the explicit identifications of truncated 11-dimensional fields with standard seven-dimensional supergravity fields.

3 Truncating the bulk theory to seven dimensions

In this section, we describe in detail how the bulk theory is truncated to seven dimensions. We recall from the introduction that this constitutes one of the essential steps in the construction of the theory. As a preparation, we explicitly write down the components of the 11-dimensional fields that survive on the orbifold plane and work out how these fit into seven-dimensional super-multiplets. We then describe, for each orbifold, the seven-dimensional EYM supergravity with the appropriate field content. By an explicit reduction of the 11-dimensional theory and comparison with this seven-dimensional theory, we find a list of identification between 11- and 7-dimensional fields which is essential for our subsequent construction.

To discuss the truncated field content, we use the representations (2.19), (2.21), (2.22) of R, T and S and the orbifold conditions (2.7)-(2.16) for y independent fields. For $N > 2$ we find that the surviving components are given by $g_{\mu\nu}$, $e_p{}^{\underline{q}}$, $C_{\mu\nu\rho}$, $C_{\mu p\bar{q}}$, $\Psi_{\mu i}$, $(\Gamma^p \Psi_p)_i$ and $(\Gamma^{\bar{p}} \Psi_{\bar{p}})_i$. Meanwhile, the spinor η which parameterizes supersymmetry reduces to η_i, a single symplectic Majorana spinor, which corresponds to seven-dimensional $\mathcal{N} = 1$ supersymmetry. Comparing with the structure of seven-dimensional multiplets (see Appendix C for a review of seven-dimensional EYM supergravity), these field components fill out the seven-dimensional supergravity multiplet and a single $U(1)$ vector multiplet. For the case of the $\mathbb{Z}_2$ orbifold, a greater field content survives, corresponding in seven-dimensions to a gravity multiplet plus three $U(1)$ vector multiplets. The surviving fields in this case are expressed most succinctly by $g_{\mu\nu}$, $e_A{}^{\underline{B}}$, $C_{\mu\nu\rho}$, $C_{\mu AB}$, $\Psi_{\mu i}$ and $\Psi_{A\bar{\imath}}$. The spinor η which parameterizes supersymmetry again reduces to η_i, a single symplectic Majorana spinor.

These results imply that the truncated bulk theory is a seven-dimensional $\mathcal{N} = 1$ EYM supergravity with gauge group $U(1)^n$, where $n = 1$ for $N > 2$ and $n = 3$ for $N = 2$. In the following, we discuss both cases and, wherever possible, use a unified description in terms of n, which can be set to either 1 or 3, as appropriate. The correspondence between 11-dimensional truncated fields and seven-dimensional supermultiplets is as follows. The gravity super-multiplet contains the purely seven-dimensional parts of the 11-dimensional metric, gravitino and three-form, that is, $g_{\mu\nu}$, $\Psi_{\mu i}$ and $C_{\mu\nu\rho}$, along with three vectors from $C_{\mu AB}$, a spinor from $\Psi_{A\bar{\imath}}$ and the scalar $\det(e_A{}^{\underline{B}})$. The remaining degrees of freedom, that is, the remaining vector(s) from $C_{\mu AB}$, the remaining spinor(s) from $\Psi_{A\bar{\imath}}$ and the scalars contained in $v_A{}^{\underline{B}} := \det(e_A{}^{\underline{B}})^{-1/4} e_A{}^{\underline{B}}$, the unit-determinant part of $e_A{}^{\underline{B}}$, fill out n seven-dimensional vector multiplets. The $n + 3$ Abelian gauge fields transform under the $SO(3, n)$ global symmetry of the $d = 7$ EYM supergravity while the vector multiplet scalars parameterize the coset $SO(3, n)/SO(3) \times SO(n)$. Let us describe how such coset spaces are obtained from the vierbein $v_A{}^{\underline{B}}$, starting with the generic case $N > 2$ with seven-dimensional gauge group $U(1)$, that is, $n = 1$. In this case, the rescaled vierbein $v_A{}^{\underline{B}}$ reduces to $v_p{}^{\underline{q}}$, which represents a set of 2×2 matrices with determinant one, identified by $SU(2)$ transformations acting on the tangent space index. Hence, these matrices form the coset $SL(2, \mathbb{C})/SU(2)$ which is isomorphic to $SO(3, 1)/SO(3)$, the correct coset space for $n = 1$. For the special $\mathbb{Z}_2$ case, which implies $n = 3$, the whole of $v_A{}^{\underline{B}}$ is present and forms the coset space $SL(4, \mathbb{R})/SO(4)$. This space is isomorphic to $SO(3, 3)/SO(3)^2$ which is indeed the correct coset space for $n = 3$.

We now briefly review seven-dimensional EYM supergravity with gauge group $U(1)^n$. A more general account of seven-dimensional supergravity including non-Abelian gauge groups can be found

in Appendix C. The seven-dimensional $\mathcal{N}=1$ supergravity multiplet contains the vielbein $\tilde{e}_\mu{}^{\underline{\nu}}$, the gravitino $\psi_{\mu i}$, a triplet of vectors $A_{\mu i}{}^j$ with field strengths $F_i{}^j = \mathrm{d}A_i{}^j$, a three-form $\tilde{C}$ with field strength $\tilde{G} = \mathrm{d}\tilde{C}$, a spinor χ_i, and a scalar σ. A seven-dimensional vector multiplet contains a $U(1)$ gauge field A_μ with field strength $F = \mathrm{d}A$, a gaugino λ_i and a triplet of scalars $\phi_i{}^j$. Here, all spinors are symplectic Majorana spinors and indices $i, j, \ldots = 1, 2$ transform under the $SU(2)$ R-symmetry. For ease of notation, the three vector fields in the supergravity multiplet and the n additional Abelian gauge fields from the vector multiplet are combined into a single $SO(3,n)$ vector A_μ^I, where $I, J, \ldots = 1, \ldots, n+3$. The coset space $SO(3,n)/SO(3)\times SO(n)$ is described by a $(3+n)\times(3+n)$ matrix $\ell_I{}^{\underline{J}}$, which depends on the $3n$ vector multiplet scalars and satisfies the $SO(3,n)$ orthogonality condition

$$\ell_I{}^{\underline{J}} \ell_K{}^{\underline{L}} \eta_{\underline{JL}} = \eta_{IK} \tag{3.1}$$

with $(\eta_{IJ}) = (\eta_{\underline{IJ}}) = \mathrm{diag}(-1,-1,-1,+1,\ldots,+1)$. Here, indices $I, J, \ldots = 1, \ldots, (n+3)$ transform under $SO(3,n)$. Their flat counterparts $\underline{I}, \underline{J} \ldots$ decompose into a triplet of $SU(2)$, corresponding to the gravitational directions and n remaining directions corresponding to the vector multiplets. Thus we can write $\ell_I{}^{\underline{J}} \to (\ell_I{}^u, \ell_I{}^\alpha)$, where $u = 1,2,3$ and $\alpha = 1, \ldots, n$. The adjoint $SU(2)$ index u can be converted into a pair of fundamental $SU(2)$ indices by multiplication with the Pauli matrices, that is,

$$\ell_I{}^i{}_j = \frac{1}{\sqrt{2}} \ell_I{}^u (\sigma_u)^i{}_j. \tag{3.2}$$

The Maurer-Cartan forms p and q of the matrix ℓ, defined by

$$p_{\mu\alpha}{}^i{}_j = \ell^I{}_\alpha \partial_\mu \ell_I{}^i{}_j, \tag{3.3}$$

$$q_\mu{}^i{}_j{}^k{}_l = \ell^{Ii}{}_j \partial_\mu \ell_I{}^k{}_l, \tag{3.4}$$

$$q_\mu{}^i{}_j = \ell^{Ii}{}_k \partial_\mu \ell_I{}^k{}_j, \tag{3.5}$$

will be needed as well.

With everything in place, we can now write down our expression for $\mathcal{L}_7^{(n)}$, the Lagrangian of seven-dimensional $\mathcal{N}=1$ EYM supergravity with gauge group $U(1)^n$ [27]. Neglecting four-fermi terms, it is given by

$$\begin{aligned}
\mathcal{L}_7^{(n)} = \frac{1}{\kappa_7^2}\sqrt{-\tilde{g}}\Big\{ & \frac{1}{2}R - \frac{1}{2}\bar{\psi}^i_\mu \Upsilon^{\mu\nu\rho}\hat{\mathcal{D}}_\nu \psi_{\rho i} - \frac{1}{4}e^{-2\sigma}\left(\ell_I{}^i{}_j \ell_J{}^j{}_i + \ell_I{}^\alpha \ell_{J\alpha}\right) F^I_{\mu\nu} F^{J\mu\nu} \\
& - \frac{1}{96} e^{4\sigma} \tilde{G}_{\mu\nu\rho\sigma} \tilde{G}^{\mu\nu\rho\sigma} - \frac{1}{2}\bar{\chi}^i \Upsilon^\mu \hat{\mathcal{D}}_\mu \chi_i - \frac{5}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{\sqrt{5}}{2}\left(\bar{\chi}^i \Upsilon^{\mu\nu}\psi_{\mu i} + \bar{\chi}^i \psi^\nu_i\right)\partial_\nu \sigma \\
& - \frac{1}{2}\bar{\lambda}^{\alpha i}\Upsilon^\mu \hat{\mathcal{D}}_\mu \lambda_{\alpha i} - \frac{1}{2} p_{\mu\alpha}{}^i{}_j p^{\mu\alpha j}{}_i - \frac{1}{\sqrt{2}}\left(\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\psi_{\mu j} + \bar{\lambda}^{\alpha i}\psi^\nu_j\right) p_{\nu\alpha}{}^j{}_i \\
& + e^{2\sigma}\tilde{G}_{\mu\nu\rho\sigma}\Big[\frac{1}{192}\left(12\bar{\psi}^{\mu i}\Upsilon^{\nu\rho}\psi^\sigma_i + \bar{\psi}^i_\lambda \Upsilon^{\lambda\mu\nu\rho\sigma\tau}\psi_{\tau i}\right) + \frac{1}{48\sqrt{5}}\left(4\bar{\chi}^i \Upsilon^{\mu\nu\rho}\psi^\sigma_i\right.
\end{aligned}$$

$$
\left. -\bar{\chi}^i\Upsilon^{\mu\nu\rho\sigma\tau}\psi_{\tau i}) - \frac{1}{320}\bar{\chi}^i\Upsilon^{\mu\nu\rho\sigma}\chi_i + \frac{1}{192}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu\rho\sigma}\lambda_{\alpha i}\right]
$$

$$
-ie^{-\sigma}F^I_{\mu\nu}\ell_I{}^j{}_i\left[\frac{1}{4\sqrt{2}}\left(\bar{\psi}^i_\rho\Upsilon^{\mu\nu\rho\sigma}\psi_{\sigma j} + 2\bar{\psi}^{\mu i}\psi^\nu_j\right) + \frac{1}{2\sqrt{10}}\left(\bar{\chi}^i\Upsilon^{\mu\nu\rho}\psi_{\rho j} - 2\bar{\chi}^i\Upsilon^\mu\psi^\nu_j\right)\right.
$$

$$
\left. + \frac{3}{20\sqrt{2}}\bar{\chi}^i\Upsilon^{\mu\nu}\chi_j - \frac{1}{4\sqrt{2}}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\lambda_{\alpha j}\right]
$$

$$
+e^{-\sigma}F^I_{\mu\nu}\ell_{I\alpha}\left[\frac{1}{4}\left(2\bar{\lambda}^{\alpha i}\Upsilon^\mu\psi^\nu_i - \bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu\rho}\psi_{\rho i}\right) + \frac{1}{2\sqrt{5}}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\chi_i\right]
$$

$$
\left. -\frac{1}{96}\epsilon^{\mu\nu\rho\sigma\kappa\lambda\tau}C_{\mu\nu\rho}F^{\tilde{I}}_{\sigma\kappa}F_{\tilde{I}\lambda\tau}\right\} . \tag{3.6}
$$

In this Lagrangian the covariant derivatives of symplectic Majorana spinors ϵ_i are defined by

$$
\hat{\mathcal{D}}_\mu\epsilon_i = \partial_\mu\epsilon_i + \frac{1}{2}q_{\mu i}{}^j\epsilon_j + \frac{1}{4}\tilde{\omega}_\mu{}^{\underline{\mu\nu}}\Upsilon_{\underline{\mu\nu}}\epsilon_i. \tag{3.7}
$$

The associated supersymmetry transformations, parameterized by the spinor ε_i, are, up to cubic fermion terms, given by

$$
\begin{aligned}
\delta\sigma &= \frac{1}{\sqrt{5}}\bar{\chi}^i\varepsilon_i , \\
\delta\tilde{e}_\mu^{\underline{\nu}} &= \bar{\varepsilon}^i\Upsilon^{\underline{\nu}}\psi_{\mu i} , \\
\delta\psi_{\mu i} &= 2\hat{\mathcal{D}}_\mu\varepsilon_i - \frac{e^{2\sigma}}{80}\left(\Upsilon_\mu{}^{\nu\rho\sigma\eta} - \frac{8}{3}\delta^\nu_\mu\Upsilon^{\rho\sigma\eta}\right)\varepsilon_i\tilde{G}_{\nu\rho\sigma\eta} + \frac{ie^{-\sigma}}{5\sqrt{2}}\left(\Upsilon_\mu{}^{\nu\rho} - 8\delta^\nu_\mu\Upsilon^\rho\right)\varepsilon_jF^I_{\nu\rho}\ell_I{}^j{}_i , \\
\delta\chi_i &= \sqrt{5}\Upsilon^\mu\varepsilon_i\partial_\mu\sigma - \frac{1}{24\sqrt{5}}\Upsilon^{\mu\upsilon\rho\sigma}\varepsilon_i\tilde{G}_{\mu\nu\rho\sigma}e^{2\sigma} - \frac{i}{\sqrt{10}}\Upsilon^{\mu\nu}\varepsilon_jF^I_{\mu\nu}\ell_I{}^j{}_ie^{-\sigma}, \\
\delta\tilde{C}_{\mu\nu\rho} &= \left(-3\bar{\psi}^i_{[\mu}\Upsilon_{\nu\rho]}\varepsilon_i - \frac{2}{\sqrt{5}}\bar{\chi}^i\Upsilon_{\mu\nu\rho}\varepsilon_i\right)e^{-2\sigma}, \\
\ell_I{}^i{}_j\delta A^I_\mu &= \left[i\sqrt{2}\left(\bar{\psi}^i_\mu\varepsilon_j - \frac{1}{2}\delta^i_j\bar{\psi}^k_\mu\varepsilon_k\right) - \frac{2i}{\sqrt{10}}\left(\bar{\chi}^i\Upsilon_\mu\varepsilon_j - \frac{1}{2}\delta^i_j\bar{\chi}^k\Upsilon_\mu\varepsilon_k\right)\right]e^\sigma, \\
\ell_I{}^\alpha\delta A^I_\mu &= \bar{\varepsilon}^i\Upsilon_\mu\lambda^\alpha_ie^\sigma, \\
\delta\ell_I{}^i{}_j &= -i\sqrt{2}\bar{\varepsilon}^i\lambda_{\alpha j}\ell_I{}^\alpha + \frac{i}{\sqrt{2}}\bar{\varepsilon}^k\lambda_{\alpha k}\ell_I{}^\alpha\delta^i_j , \\
\delta\ell_I{}^\alpha &= -i\sqrt{2}\bar{\varepsilon}^i\lambda^\alpha_j\ell_I{}^j{}_i , \\
\delta\lambda^\alpha_i &= -\frac{1}{2}\Upsilon^{\mu\nu}\varepsilon_iF^I_{\mu\nu}\ell_I{}^\alpha e^{-\sigma} + \sqrt{2}i\Upsilon^\mu\varepsilon_jp_\mu{}^{\alpha i}{}_j .
\end{aligned} \tag{3.8}
$$

We now explain in detail how the truncated bulk theory corresponds to the above seven-dimensional EYM supergravity with gauge group $U(1)^n$, where $n = 1$ for the $\mathbb{Z}_N$ orbifold with $N > 2$ and $n = 3$ for the special $\mathbb{Z}_2$ case. It is convenient to choose the seven-dimensional Newton constant κ_7 as $\kappa_7 = \kappa^{5/9}$. The correspondence between 11- and 7-dimensional Lagrangians can then be written as

$$
\kappa^{8/9}\mathcal{L}_{11}|_{y=0} = \mathcal{L}_7^{(n)} . \tag{3.9}
$$

We have verified by explicit computation that this relation indeed holds for appropriate identifications of the truncated 11-dimensional fields with the standard seven-dimensional fields which appear in Eq. (3.6). For the generic $\mathbb{Z}_N$ orbifold with $N > 2$ and $n = 1$, they are given by

$$
\begin{aligned}
\sigma &= \frac{3}{20}\ln\det g_{AB}, &(3.10)\\
\tilde{g}_{\mu\nu} &= e^{\frac{4}{3}\sigma} g_{\mu\nu}, &(3.11)\\
\psi_{\mu i} &= \Psi_{\mu i} e^{\frac{1}{3}\sigma} - \frac{1}{5}\Upsilon_\mu \left(\Gamma^A \Psi_A\right)_i e^{-\frac{1}{3}\sigma}, &(3.12)\\
\tilde{C}_{\mu\nu\rho} &= C_{\mu\nu\rho}, &(3.13)\\
\chi_i &= \frac{3}{2\sqrt{5}} \left(\Gamma^A \Psi_A\right)_i e^{-\frac{1}{3}\sigma}, &(3.14)\\
F^I_{\mu\nu} &= -\frac{i}{2}\mathrm{tr}\left(\sigma^I G_{\mu\nu}\right), &(3.15)\\
\lambda_i &= \frac{i}{2}\left(\Gamma^p \Psi_p - \Gamma^{\bar{p}} \Psi_{\bar{p}}\right)_i e^{-\frac{1}{3}\sigma}, &(3.16)\\
\ell_I{}^{\underline{J}} &= \frac{1}{2}\mathrm{tr}\left(\bar{\sigma}_I v \sigma^J v^\dagger\right). &(3.17)
\end{aligned}
$$

Furthermore the seven-dimensional supersymmetry spinor ε_i is related to its 11-dimensional counterpart η by

$$\varepsilon_i = e^{\frac{1}{3}\sigma}\eta_i. \tag{3.18}$$

In these identities, we have defined the matrices $G_{\mu\nu} \equiv (G_{\mu\nu p\bar{q}})$, $v \equiv (e^{5\sigma/6} e^{\bar{p}}{}_{\underline{\bar{q}}})$ and made use of the standard $SO(3,1)$ Pauli matrices σ^I, defined in Appendix B. For the $\mathbb{Z}_2$ orbifold we have $n = 3$ and, therefore, two additional $U(1)$ vector multiplets. Not surprisingly, field identification in the gravity multiplet sector is unchanged from the generic case and still given by Eqs. (3.10)-(3.14). It is in the vector multiplet sector, where the additional states appear, that we have to make a distinction. For the bosonic vector multiplet fields we find

$$
\begin{aligned}
F^I_{\mu\nu} &= -\frac{1}{4}\mathrm{tr}\left(T^I G_{\mu\nu}\right), &(3.19)\\
\ell_I{}^{\underline{J}} &= \frac{1}{4}\mathrm{tr}\left(\bar{T}_I v T^J v^T\right), &(3.20)
\end{aligned}
$$

where now $G_{\mu\nu} \equiv (G_{\mu\nu AB})$ and $v \equiv (e^{5\sigma/6} e^A{}_{\underline{B}})$. Here, T^I are the six $SO(4)$ generators, which are explicitly given in Appendix B.

4 General form of the supersymmetric bulk-brane action

In this section, we present our general method of construction for the full action, which combines 11-dimensional supergravity with the seven-dimensional super-Yang-Mills theory on the orbifold plane in a supersymmetric way. Main players in this construction will be the constrained 11-dimensional bulk theory $\mathcal{L}_{11}$, as discussed in Section 2, its truncation to seven dimensions, $\mathcal{L}_7^{(n)}$, which has been discussed in the previous section and corresponds to a $d = 7$ EYM supergravity with gauge group

$U(1)^n$ and $\mathcal{L}_{SU(N)}$, a $d = 7$ EYM supergravity with gauge group $U(1)^n \times SU(N)$. The $SU(N)$ gauge group in the latter theory corresponds, of course, to the additional $SU(N)$ gauge multiplet which one expects to arise for M-theory on the orbifold $\mathbb{C}^2/\mathbb{Z}_N$.

Let us briefly discuss the physical origin of these $SU(N)$ gauge fields on the orbifold fixed plane. It is well-known [2, 28], that the $N-1$ Abelian $U(1)$ gauge fields within $SU(N)$ are already massless for a smooth blow-up of the orbifold $\mathbb{C}^2/\mathbb{Z}_N$ by an ALE manifold. More precisely, they arise as zero modes of the M-theory three-form on the blow-up ALE manifold. The remaining vector fields arise from membranes wrapping the two-cycles of the ALE space and become massless only in the singular orbifold limit, when these two-cycles collapse. For our purposes, the only relevant fact is that all these $SU(N)$ vector fields are located on the orbifold fixed plane. This allows us to treat the Abelian and non-Abelian parts of $SU(N)$ on the same footing, despite their different physical origin.

We claim that the action for the bulk-brane system is given by

$$S_{11-7} = \int_{\mathcal{M}_{11}^N} \mathrm{d}^{11}x \left[\mathcal{L}_{11} + \delta^{(4)}(y^A)\mathcal{L}_{\mathrm{brane}}\right], \tag{4.1}$$

where

$$\mathcal{L}_{\mathrm{brane}} = \mathcal{L}_{\mathrm{SU(N)}} - \mathcal{L}_7^{(n)}. \tag{4.2}$$

Here, as before, $\mathcal{L}_{11}$ is the Lagrangian for 11-dimensional supergravity (2.2) with fields constrained in accordance with the orbifold $\mathbb{Z}_N$ symmetry, as discussed in Section 2. The Lagrangian $\mathcal{L}_7^{(n)}$ describes a seven-dimensional $\mathcal{N} = 1$ EYM theory with gauge group $U(1)^n$. Choosing $n = 1$ for generic $\mathbb{Z}_N$ with $N > 2$ and $n = 3$ for $\mathbb{Z}_2$, this Lagrangian corresponds to the truncation of the bulk Lagrangian $\mathcal{L}_{11}$ to seven dimensions, as we have shown in the previous section. This correspondence implies identifications between truncated 11-dimensional bulk fields and the fields in $\mathcal{L}_7^{(n)}$, which have also been worked out explicitly in the previous section (see Eqs. (3.10)–(3.17) for the case $N > 2$ and Eqs. (3.10)–(3.14) and (3.19)–(3.20) for $N = 2$). These identifications are also considered part of the definition of the Lagrangian (4.1). The new Lagrangian $\mathcal{L}_{SU(N)}$ is that of seven-dimensional EYM supergravity with gauge group $U(1)^n \times SU(N)$, where, as usual, $n = 1$ for generic $\mathbb{Z}_N$ with $N > 2$ and $n = 3$ for $\mathbb{Z}_2$. This Lagrangian contains the "old" states in the gravity multiplet and the $U(1)^n$ gauge multiplet and the "new" states in the $SU(N)$ gauge multiplet. We will think of the former states as being identified with the truncated 11-dimensional bulk states by precisely the same relations we have used for $\mathcal{L}_7^{(n)}$. The idea of this construction is, of course, that in $\mathcal{L}_{\mathrm{brane}}$ the pure supergravity and $U(1)^n$ vector multiplet parts cancel between $\mathcal{L}_{SU(N)}$ and $\mathcal{L}_7^{(n)}$, so that we remain with "pure" $SU(N)$ theory on the orbifold plane. For this to work out, we have to choose the seven-dimensional Newton constant κ_7 within $\mathcal{L}_{SU(N)}$ to be the same as the one in $\mathcal{L}_7^{(n)}$, that is

$$\kappa_7 = \kappa^{5/9}. \tag{4.3}$$

The supersymmetry transformation laws for the action (4.1) are schematically given by

$$\delta_{11} = {\delta_{11}}^{11} + \kappa^{8/9}\delta^{(4)}(y^A){\delta_{11}}^{\mathrm{brane}}, \tag{4.4}$$

$$\delta_7 = {\delta_7}^{\mathrm{SU(N)}}, \tag{4.5}$$

where

$$\delta_{11}{}^{\text{brane}} = \delta_{11}{}^{\text{SU(N)}} - \delta_{11}{}^{11}. \tag{4.6}$$

Here δ_{11} and δ_7 denote the supersymmetry transformation of bulk fields and fields on the orbifold fixed plane, respectively. A superscript 11 indicates a supersymmetry transformation law of $\mathcal{L}_{11}$, as given in equations (2.4), and a superscript $SU(N)$ indicates a supersymmetry transformation law of $\mathcal{L}_{SU(N)}$, as can be found by substituting the appropriate gauge group into equations (C.16). These transformation laws are parameterized by a single 11-dimensional spinor, with the seven-dimensional spinors in $\delta_{11}{}^{\text{SU(N)}}$ and $\delta_7{}^{\text{SU(N)}}$ being simply related to this 11-dimensional spinor by equation (3.18). On varying S_{11-7} with respect to these supersymmetry transformations we find

$$\delta S_{11-7} \;\; = \;\; -\int_{\mathcal{M}_{11}} \mathrm{d}^{11}x \delta^{(4)}(y^A) \left(1 - \kappa^{8/9}\delta^{(4)}(y^A)\right) \delta_{11}{}^{\text{brane}} \mathcal{L}_{\text{brane}}. \tag{4.7}$$

At first glance, the occurrence of delta-function squared terms is concerning. However, as in Hořava-Witten theory [3], we can interpret the occurrence of these terms as a symptom of attempting to treat in classical supergravity what really should be treated in quantum M-theory. It is presumed that in proper quantum M-theory, fields on the brane penetrate a finite thickness into the bulk, and that there would be some kind of built-in cutoff allowing us to replace $\delta^{(4)}(0)$ by a finite constant times $\kappa^{-8/9}$. If we could set this constant to one and formally substitute

$$\delta^{(4)}(0) = \kappa^{-8/9} \tag{4.8}$$

then the above integral would vanish.

As in Ref. [3], we can avoid such a regularization if we work only to lowest non-trivial order in κ, or, more precisely to lowest non-trivial order in the parameter $h = \kappa_7/g_{\text{YM}}$. Note that h has dimension of inverse energy. To determine the order in h of various terms in the Lagrangian we need to fix a convention for the energy dimensions of the fields. We assign energy dimension 0 to bulk bosonic fields and energy dimension 1/2 to bulk fermions. This is consistent with the way we have written down 11-dimensional supergravity (2.2). In terms of seven-dimensional supermultiplets this tells us to assign energy dimension 0 to the gravity multiplet and the $U(1)$ vector multiplet bosons and energy dimension 1/2 to the fermions in these multiplets. For the $SU(N)$ vector multiplet, that is for the brane fields, we assign energy dimension 1 to the bosons and 3/2 to the fermions. With these conventions we can expand

$$\mathcal{L}_{SU(N)} = \kappa_7^{-2} \left(\mathcal{L}_{(0)} + h^2\mathcal{L}_{(2)} + h^4\mathcal{L}_{(4)} + \ldots\right), \tag{4.9}$$

where the $\mathcal{L}_{(m)}$, $m = 0, 2, 4, \ldots$ are independent of h. The first term in this series is equal to $\mathcal{L}_7^{(n)}$, and therefore the leading order contribution to $\mathcal{L}_{\text{brane}}$ is precisely the second term of order h^2 in the above series. It turns out that, up to this order, the action S_{11-7} is supersymmetric under (4.4) and (4.5). To see this we also expand the supersymmetry transformation in orders of h, that is

$$\delta_{11}{}^{SU(N)} = \delta_{11}^{(0)} + h^2\delta_{11}^{(2)} + h^4\delta_{11}^{(4)} + \ldots . \tag{4.10}$$

Using this expansion and Eq. (4.9) one finds that the uncanceled variation (4.7) is, in fact, of order h^4. This means the action (4.1) is indeed supersymmetric up to order h^2 and can be used to write

down a supersymmetric theory to this order. This is exactly what we will do in the following section. We have also checked explicitly that the terms of order h^4 in Eq. (4.7) are non-vanishing, so that our method cannot be extended straightforwardly to higher orders.

A final remark concerns the value of the Yang-Mills gauge coupling $g_{\rm YM}$. The above construction does not fix the value of this coupling and our action is supersymmetric to order h^2 for all values of $g_{\rm YM}$. However, within M-theory one expects $g_{\rm YM}$ to be fixed in terms of the 11-dimensional Newton constant κ. Indeed, reducing M-theory on a circle to IIA string theory, the orbifold seven-planes turn into D6 branes whose tension is fixed in terms of the string tension [29]. By a straightforward matching procedure this fixes the gauge coupling to be [16]

$$g_{\rm YM}^2 = (4\pi)^{4/3}\kappa^{2/3} \; . \tag{4.11}$$

5 The explicit bulk/brane theory

In this section, we give a detailed description of M-theory on $\mathcal{M}_{11}^N = \mathbb{R}^{1,6} \times \mathbb{C}^2/\mathbb{Z}_N$, taking account of the additional states that appear on the brane. We begin with a reminder of how the bulk fields, truncated to seven dimensions, are identified with the fields that appear in the seven-dimensional supergravity Lagrangians from which the theory is constructed. Then we write down our full Lagrangian, and present the supersymmetry transformation laws.

As discussed in the previous section, the full Lagrangian is constructed from three parts, the Lagrangian of 11-dimensional supergravity $\mathcal{L}_{11}$ with bulk fields constrained by the orbifold action, $\mathcal{L}_7^{(n)}$, the Lagrangian of seven-dimensional EYM supergravity with gauge group $U(1)^n$ and $\mathcal{L}_{SU(N)}$, the Lagrangians for seven-dimensional EYM supergravity with gauge group $U(1)^n \times SU(N)$. The Lagrangian $\mathcal{L}_{11}$ has been written down and discussed in Section 2, whilst $\mathcal{L}_7^{(n)}$ has been dealt with in Section 3. The final piece, $\mathcal{L}_{SU(N)}$, is discussed in Appendix C, where we provide the reader with a general review of seven-dimensional supergravity theories. Crucial to our construction is the way in which we identify the fields in the supergravity and $U(1)^n$ gauge multiplets of the latter two Lagrangians with the truncated bulk fields. Let us recall the structure of this identification which has been worked out in Section 3. The bulk fields truncated to seven dimensions form a $d = 7$ gravity multiplet and n $U(1)$ vector multiplets, where $n = 1$ for the general $\mathbb{Z}_N$ orbifold with $N > 2$ and $n = 3$ for the $\mathbb{Z}_2$ orbifold. The gravity multiplet contains the purely seven-dimensional parts of the 11-dimensional metric, gravitino and three-form, that is, $g_{\mu\nu}$, $\Psi_{\mu i}$ and $C_{\mu\nu\rho}$, along with three vectors from $C_{\mu AB}$, a spinor from $\Psi_{A\bar{\imath}}$ and the scalar $\det(e_{\underline{A}}{}^{\underline{B}})$. Meanwhile, the vector multiplets contain the remaining vectors from $C_{\mu AB}$, the remaining spinors from $\Psi_{A\bar{\imath}}$ and the scalars contained in $v_{\underline{A}}{}^{\underline{B}} := \det(e_{\underline{A}}{}^{\underline{B}})^{-1/4} e_{\underline{A}}{}^{\underline{B}}$, the unit-determinant part of $e_{\underline{A}}{}^{\underline{B}}$. The gravity and $U(1)$ vector fields naturally combine together into a single entity A_μ^I, $I = 1, \ldots (n+3)$, where the index I transforms tensorially under a global $SO(3,n)$ symmetry. Meanwhile, the vector multiplet scalars naturally combine into a single $(3+n)\times(3+n)$ matrix ℓ which parameterizes the coset $SO(3,n)/SO(3)\times SO(n)$. The precise mathematical form of these identifications is given in equations (3.10)-(3.17) for the general $\mathbb{Z}_N$ orbifold with $N > 2$, and equations (3.10)-(3.14) and (3.19)-(3.20) for the $\mathbb{Z}_2$ orbifold.

In addition to those states which arise from projecting bulk states to the orbifold fixed plane the Lagrangian $\mathcal{L}_{SU(N)}$ also contains a seven-dimensional $SU(N)$ vector multiplet, which is genuinely located on the orbifold plane. It consists of gauge fields A_μ^a with field strengths $F^a = \mathcal{D}A^a$, gauginos

λ^a_i, and $SU(2)$ triplets of scalars $\phi_a{}^i{}_j$. These fields are in the adjoint of $SU(N)$ and we use $a, b, \ldots = 4, \ldots, (N^2+2)$ for $su(N)$ Lie algebra indices. It is important to write $\mathcal{L}_{SU(N)}$ in a form where the $SU(N)$ states and the gravity/$U(1)^n$ states are disentangled, since the latter must be identified with truncated bulk states, as described above. For most of the fields appearing in $\mathcal{L}_{SU(N)}$, this is just a trivial matter of using the appropriate notation. For example, the vector fields in $\mathcal{L}_{SU(N)}$ which naturally combine into a single entity $A^{\tilde{I}}_\mu$, where $\tilde{I} = 1, \ldots, (3+n+N^2-1)$, and transforms as a vector under the global $SO(3, n+N^2-1)$ symmetry, can simply be decomposed as $A^{\tilde{I}}_\mu = (A^I_\mu, A^a_\mu)$, where A^I_μ refers to the three vector fields in the gravity multiplet and the $U(1)^n$ vector fields and A^a_μ denotes the $SU(N)$ vector fields. For gauge group $U(1)^n \times SU(N)$, the associated scalar fields parameterize the coset $SO(3, n+N^2-1)/SO(3) \times SO(n+N^2-1)$. We obtain representatives L for this coset by expanding around the bulk scalar coset $SO(3,n)/SO(3) \times SO(n)$, represented by matrices ℓ, to second order in the $SU(N)$ scalars $\Phi \equiv (\phi_a{}^u)$. For the details see Appendix C.2. This leads to

$$L = \begin{pmatrix} \ell + \frac{1}{2}h^2 \ell \Phi^T \Phi & m & h\ell\Phi^T \\ h\Phi & 0 & \mathbf{1}_{N^2-1} + \frac{1}{2}h^2 \Phi\Phi^T \end{pmatrix} . \tag{5.1}$$

We note that the neglected Φ terms are of order h^3 and higher and, since we are aiming to construct the action only up to terms of order h^2, are, therefore, not relevant in the present context.

We are now ready to write down our final action. As discussed in Section 4, to order $h^2 \sim g_{\rm YM}^{-2}$, it is given by

$$S_{11-7} = \int_{\mathcal{M}_{11}^N} \mathrm{d}^{11}x \left[\mathcal{L}_{11} + \delta^{(4)}(y^A)\mathcal{L}_{\rm brane}\right] , \tag{5.2}$$

where

$$\mathcal{L}_{\rm brane} = \mathcal{L}_{\rm SU(N)} - \mathcal{L}_7^{(n)} , \tag{5.3}$$

and $n = 3$ for the $\mathbb{Z}_2$ orbifold and $n = 1$ for $\mathbb{Z}_N$ with $N > 2$. The bulk contribution, $\mathcal{L}_{11}$, is given in equation (2.2), with bulk fields subject to the orbifold constraints (2.7)–(2.16). On the orbifold fixed plane, $\mathcal{L}_7^{(n)}$ acts to cancel all the terms in $\mathcal{L}_{\rm SU(N)}$ that only depend on bulk fields projected to the orbifold plane. Thus none of the bulk gravity terms are replicated on the orbifold place. To find $\mathcal{L}_{\rm brane}$ explicitly we need to expand $\mathcal{L}_{SU(N)}$ in powers of h, using, in particular, the above expressions for the gauge fields $A^{\tilde{I}}_\mu$ and the coset matrices L, and extract the terms of order h^2. The further details of this calculation are provided in Appendix C. The result is

$$\begin{aligned}
\mathcal{L}_{\rm brane} \;=\; \frac{1}{g_{\rm YM}^2}\sqrt{-\tilde{g}}\Big\{ & -\frac{1}{4}e^{-2\sigma}F^a_{\mu\nu}F_a^{\mu\nu} - \frac{1}{2}\hat{\mathcal{D}}_\mu \phi_a{}^i{}_j \hat{\mathcal{D}}^\mu \phi^{aj}{}_i - \frac{1}{2}\bar{\lambda}^{ai}\Upsilon^\mu \hat{\mathcal{D}}_\mu \lambda_{ai} - e^{-2\sigma}\ell_I{}^i{}_j \phi_a{}^j{}_i F^I_{\mu\nu} F^{a\mu\nu} \\
& -\frac{1}{2}e^{-2\sigma}\ell_I{}^i{}_j \phi^{aj}{}_i \ell_J{}^k{}_l \phi_a{}^l{}_k F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}p_{\mu\alpha}{}^i{}_j \phi_a{}^j{}_i p^{\mu\alpha k}{}_l \phi^{al}{}_k \\
& +\frac{1}{4}\phi_a{}^i{}_k \hat{\mathcal{D}}_\mu \phi^{ak}{}_j \left(\bar{\psi}^j_\nu \Upsilon^{\nu\mu\rho}\psi_{\rho i} + \bar{\chi}^j \Upsilon^\mu \chi_i + \bar{\lambda}^{\alpha j}\Upsilon^\mu \lambda_{\alpha i}\right) \\
& -\frac{1}{2\sqrt{2}}\left(\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\psi_{\mu j} + \bar{\lambda}^{\alpha i}\psi^\nu_j\right)\phi_a{}^j{}_i \phi^{ak}{}_l p_{\nu\alpha}{}^l{}_k - \frac{1}{\sqrt{2}}\left(\bar{\lambda}^{ai}\Upsilon^{\mu\nu}\psi_{\mu j} + \bar{\lambda}^{ai}\psi^\nu_j\right)\hat{\mathcal{D}}_\nu \phi_a{}^j{}_i
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{192}e^{2\sigma}\tilde{G}_{\mu\nu\rho\sigma}\bar{\lambda}^{ai}\Upsilon^{\mu\nu\rho\sigma}\lambda_{ai}+\frac{i}{4\sqrt{2}}e^{-\sigma}F^I_{\mu\nu}\ell_I{}^j{}_i\bar{\lambda}^{ai}\Upsilon^{\mu\nu}\lambda_{aj}\\
&-\frac{i}{2}e^{-\sigma}\left(F^I_{\mu\nu}\ell_I{}^k{}_l\phi^{al}{}_k\phi_a{}^j{}_i+2F^a_{\mu\nu}\phi_a{}^j{}_i\right)\left[\frac{1}{4\sqrt{2}}\left(\bar{\psi}^i_\rho\Upsilon^{\mu\nu\rho\sigma}\psi_{\sigma j}+2\bar{\psi}^{\mu i}\psi^\nu_j\right)\right.\\
&\qquad\left.+\frac{3}{20\sqrt{2}}\bar{\chi}^i\Upsilon^{\mu\nu}\chi_j-\frac{1}{4\sqrt{2}}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\lambda_{\alpha j}+\frac{1}{2\sqrt{10}}\left(\bar{\chi}^i\Upsilon^{\mu\nu\rho}\psi_{\rho j}-2\bar{\chi}^i\Upsilon^\mu\psi^\nu_j\right)\right]\\
&+e^{-\sigma}F_{a\mu\nu}\left[\frac{1}{4}\left(2\bar{\lambda}^{ai}\Upsilon^\mu\psi^\nu_i-\bar{\lambda}^{ai}\Upsilon^{\mu\nu\rho}\psi_{\rho i}\right)+\frac{1}{2\sqrt{5}}\bar{\lambda}^{ai}\Upsilon^{\mu\nu}\chi_i\right]\\
&+\frac{1}{4}e^{2\sigma}f_{bc}{}^a f_{dea}\phi^{bi}{}_k\phi^{ck}{}_j\phi^{dj}{}_l\phi^{el}{}_i-\frac{1}{2}e^\sigma f_{abc}\phi^{bi}{}_k\phi^{ck}{}_j\left(\bar{\psi}^j_\mu\Upsilon^\mu\lambda^a_i+\frac{2}{\sqrt{5}}\bar{\chi}^j\lambda_i{}^a\right)\\
&-\frac{i}{\sqrt{2}}e^\sigma f_{ab}{}^c\phi_c{}^i{}_j\bar{\lambda}^{aj}\lambda^b_i+\frac{i}{60\sqrt{2}}e^\sigma f_{ab}{}^c\phi^{al}{}_k\phi^{bj}{}_l\phi_c{}^k{}_j\left(5\bar{\psi}^i_\mu\Upsilon^{\mu\nu}\psi_{\nu i}+2\sqrt{5}\bar{\psi}^i_\mu\Upsilon^\mu\chi_i\right.\\
&\qquad\left.\left.+3\bar{\chi}^i\chi_i-5\bar{\lambda}^{\alpha i}\lambda_{\alpha i}\right)-\frac{1}{96}\epsilon^{\mu\nu\rho\sigma\kappa\lambda\tau}\tilde{C}_{\mu\nu\rho}F^a_{\sigma\kappa}F_{a\lambda\tau}\right\}. \qquad (5.4)
\end{aligned}
$$

Here $f_{ab}{}^c$ are the structure constants of $SU(N)$. The covariant derivatives that appear are given by

$$
\begin{aligned}
\mathcal{D}_\mu A_{\nu a} &= \partial_\mu A_{\nu a}-\tilde{\Gamma}^\rho_{\mu\nu}A_{\rho a}+f_{ab}{}^c A^b_\mu A^c_\nu, &(5.5)\\
\hat{\mathcal{D}}_\mu\lambda_{ai} &= \partial_\mu\lambda_{ai}+\frac{1}{2}q_{\mu i}{}^j\lambda_{aj}+\frac{1}{4}\tilde{\omega}_\mu{}^{\underline{\mu\nu}}\Upsilon_{\underline{\mu\nu}}\lambda_{ai}+f_{ab}{}^c A^b_\mu\lambda_{ci}, &(5.6)\\
\hat{\mathcal{D}}_\mu\phi_a{}^i{}_j &= \partial_\mu\phi_a{}^i{}_j-q_\mu{}^i{}_j{}^k{}_l\phi_a{}^l{}_k+f_{ab}{}^c A^b_\mu\phi_c{}^i{}_j, &(5.7)
\end{aligned}
$$

with the Christoffel and spin connections $\tilde{\Gamma}$ and $\tilde{\omega}$ taken in the seven-dimensional Einstein frame, (with respect to the metric $\tilde{g}$). Finally, the quantities p and q are the Maurer-Cartan forms of the bulk scalar coset matrix $\ell_I{}^{\underline{J}}$ as given by equations (3.3)–(3.5). Once again, the identities for relating the seven-dimensional gravity and $U(1)$ vector multiplet fields to 11-dimensional bulk fields are given in equations (3.10)–(3.17) for the generic $\mathbb{Z}_N$ orbifold with $N>2$, and equations (3.10)–(3.14) and (3.19)–(3.20) for the $\mathbb{Z}_2$ orbifold. We stress that these identifications are part of the definition of the theory.

The leading order brane corrections to the supersymmetry transformation laws (2.4) of the bulk fields are computed using equations (4.4) and (4.5). They are given by

$$
\begin{aligned}
\delta^{\rm brane}\psi_{\mu i} &= \frac{\kappa_7^2}{g^2_{\rm YM}}\left\{\frac{1}{2}\left(\phi_{ak}{}^j\hat{\mathcal{D}}_\mu\phi^a{}_i{}^k-\phi^a{}_i{}^k\hat{\mathcal{D}}_\mu\phi_{ak}{}^j\right)\varepsilon_j-\frac{i}{15\sqrt{2}}\Upsilon_\mu\varepsilon_i f_{ab}{}^c\phi^{al}{}_k\phi^{bj}{}_l\phi_c{}^k{}_j e^\sigma\right.\\
&\qquad\left.+\frac{i}{10\sqrt{2}}\left(\Upsilon_\mu{}^{\nu\rho}-8\delta^\nu_\mu\Upsilon^\rho\right)\varepsilon_j\left(F^I_{\nu\rho}\ell_I{}^k{}_l\phi^{al}{}_k\phi_a{}^j{}_i+2F^a_{\nu\rho}\phi_a{}^j{}_i\right)e^{-\sigma}\right\},\\
\delta^{\rm brane}\chi_i &= \frac{\kappa_7^2}{g^2_{\rm YM}}\left\{-\frac{i}{2\sqrt{10}}\Upsilon^{\mu\nu}\varepsilon_j\left(F^I_{\mu\nu}\ell_I{}^k{}_l\phi^{al}{}_k\phi_a{}^j{}_i+2F^a_{\mu\nu}\phi_a{}^j{}_i\right)e^{-\sigma}\right.\\
&\qquad\left.+\frac{i}{3\sqrt{10}}\varepsilon_i f_{ab}{}^c\phi^{al}{}_k\phi^{bj}{}_l\phi_c{}^k{}_j e^\sigma\right\},\\
\ell_I{}^i{}_j\delta^{\rm brane}A^I_\mu &= \frac{\kappa_7^2}{g^2_{\rm YM}}\left\{\left(\frac{i}{\sqrt{2}}\bar{\psi}^k_\mu\varepsilon_l-\frac{i}{\sqrt{10}}\bar{\chi}^k\Upsilon_\mu\varepsilon_l\right)\phi^{al}{}_k\phi_a{}^i{}_j e^\sigma-\bar{\varepsilon}^k\Upsilon_\mu\lambda^a_k\phi_a{}^i{}_j e^\sigma\right\}, \qquad (5.8)
\end{aligned}
$$

$$
\begin{aligned}
\ell_I{}^\alpha \delta^{\rm brane} A^I_\mu &= 0, \\
\delta^{\rm brane} \ell_I{}^i{}_j &= \frac{\kappa_7^2}{g^2_{\rm YM}} \left\{ \frac{i}{\sqrt{2}} \left[\bar{\varepsilon}^k \lambda_{\alpha l} \phi^{al}{}_k \phi_a{}^i{}_j \ell_I{}^\alpha + \bar{\varepsilon}^l \lambda_{ak} \phi^{ai}{}_j \ell_I{}^k{}_l - \left(\bar{\varepsilon}^i \lambda_{aj} - \frac{1}{2} \delta^i_j \bar{\varepsilon}^m \lambda_{am} \right) \phi^{al}{}_k \ell_I{}^k{}_l \right] \right\}, \\
\delta^{\rm brane} \ell_I{}^\alpha &= \frac{\kappa_7^2}{g^2_{\rm YM}} \left\{ - \frac{i}{\sqrt{2}} \bar{\varepsilon}^i \lambda^\alpha_j \phi^{aj}{}_i \phi_a{}^l{}_k \ell_I{}^k{}_l \right\}, \\
\delta^{\rm brane} \lambda^\alpha_i &= \frac{\kappa_7^2}{g^2_{\rm YM}} \left\{ \frac{i}{\sqrt{2}} \Upsilon^\mu \varepsilon_j \phi_{ai}{}^j p_\mu{}^{\alpha k}{}_l \phi^{al}{}_k \right\},
\end{aligned}
$$

where ε_i is the 11-dimensional supersymmetry spinor η projected onto the orbifold plane, as given in (3.18). We note that not all of the bulk fields receive corrections to their supersymmetry transformation laws. The leading order supersymmetry transformation laws of the $SU(N)$ multiplet fields are found using equation (4.5) and take the form

$$
\begin{aligned}
\delta A^a_\mu &= \bar{\varepsilon}^i \Upsilon_\mu \lambda^a_i e^\sigma - \left(i\sqrt{2} \psi^i_\mu \varepsilon_j - \frac{2i}{\sqrt{10}} \bar{\chi}^i \Upsilon_\mu \varepsilon_j \right) \phi^{aj}{}_i e^\sigma, \\
\delta \phi_a{}^i{}_j &= -i\sqrt{2} \left(\bar{\varepsilon}^i \lambda_{aj} - \frac{1}{2} \delta^i_j \bar{\varepsilon}^k \lambda_{ak} \right), \\
\delta \lambda^a_i &= -\frac{1}{2} \Upsilon^{\mu\nu} \varepsilon_i \left(F^I_{\mu\nu} \ell_I{}^j{}_k \phi^{ak}{}_j + F^a_{\mu\nu} \right) e^{-\sigma} - i\sqrt{2} \Upsilon^\mu \varepsilon_j \hat{\mathcal{D}}_\mu \phi^a{}_i{}^j - i\varepsilon_j f^a{}_{bc} \phi^{bj}{}_k \phi^{ck}{}_i.
\end{aligned}
\tag{5.9}
$$

To make some of the properties of our result more transparent, it is helpful to extract the bosonic part of the action. This bosonic part will also be sufficient for many practical applications. We recall that the full Lagrangian (5.4) is written in the seven-dimensional Einstein frame to avoid the appearance of σ–dependent pre-factors in many terms. The bosonic part, however, can be conveniently formulated in terms of $g_{\mu\nu}$, the seven-dimensional part of the 11-dimensional bulk metric g_{MN}. This requires performing the Weyl-rescaling (3.11). It also simplifies the notation if we rescale the scalar σ as $\tau = 10\sigma/3$, and drop the tilde from the three-form $\tilde{C}_{\mu\nu\rho}$ and its field strength $\tilde{G}_{\mu\nu\rho\sigma}$, which exactly coincide with the purely seven-dimensional components of their 11-dimensional counterparts. Let us now write down the purely bosonic part of our action, subject to these small modifications. We find

$$
\mathcal{S}_{11-7,{\rm bos}} = \mathcal{S}_{11,{\rm bos}} + \mathcal{S}_{7,{\rm bos}}\,, \tag{5.10}
$$

where $\mathcal{S}_{11,{\rm bos}}$ is the bosonic part of 11-dimensional supergravity (2.2), with fields subject to the orbifold constraints (2.7)–(2.12). Further, $\mathcal{S}_{7,{\rm bos}}$ is the bosonic part of Eq. (5.4), subject to the above modifications, for which we obtain

$$
\begin{aligned}
\mathcal{S}_{7,{\rm bos}} &= \frac{1}{g^2_{\rm YM}} \int_{y=0} {\rm d}^7 x \sqrt{-g} \left(-\frac{1}{4} H_{ab} F^a_{\mu\nu} F^{b\mu\nu} - \frac{1}{2} H_{aI} F^a_{\mu\nu} F^{I\mu\nu} - \frac{1}{4} (\delta H)_{IJ} F^I_{\mu\nu} F^{J\mu\nu} \right. \\
&\qquad \left. - \frac{1}{2} e^\tau \hat{\mathcal{D}}_\mu \phi_a{}^i{}_j \hat{\mathcal{D}}^\mu \phi^{aj}{}_i - \frac{1}{2} (\delta K)^{\alpha j}{}_i{}^{\beta l}{}_k \, p_{\mu\alpha}{}^i{}_j p^\mu{}_\beta{}^k{}_l + \frac{1}{4} D^{ai}{}_j D_a{}^j{}_i \right) \\
&\quad - \frac{1}{4 g^2_{\rm YM}} \int_{y=0} C \wedge F^a \wedge F_a,
\end{aligned}
\tag{5.11}
$$

where

$$\begin{aligned} H_{ab} &= \delta_{ab}, && (5.12)\\ H_{aI} &= 2\ell_I{}^i{}_j\phi_a{}^j{}_i, && (5.13)\\ (\delta H)_{IJ} &= 2\ell_I{}^i{}_j\phi_a{}^j{}_i\ell_J{}^k{}_l\phi_a{}^l{}_k, && (5.14)\\ (\delta K)^{\alpha j}{}_i{}^{\beta l}{}_k &= e^\tau\delta^{\alpha\beta}\phi_a{}^j{}_i\phi^{al}{}_k, && (5.15)\\ D^{ai}{}_j &= e^\tau f^a{}_{bc}\phi^{bi}{}_k\phi^{ck}{}_j. && (5.16) \end{aligned}$$

The gauge covariant derivative is denoted by $\mathcal{D}$, whilst $\hat{\mathcal{D}}$ is given by

$$\hat{\mathcal{D}}_\mu\phi_a{}^i{}_j = \mathcal{D}_\mu\phi_a{}^i{}_j - q_\mu{}^i{}_j{}^k{}_l\phi_a{}^l{}_k\,. \tag{5.17}$$

The Maurer-Cartan forms p and q of the matrix of scalars ℓ are defined by

$$\begin{aligned} p_{\mu\alpha}{}^i{}_j &= \ell^I{}_\alpha\partial_\mu\ell_I{}^i{}_j, && (5.18)\\ q_\mu{}^i{}_j{}^k{}_l &= \ell^{Ii}{}_j\partial_\mu\ell_I{}^k{}_l. && (5.19) \end{aligned}$$

The bosonic fields localized on the orbifold plane are the $SU(N)$ gauge vectors $F^a = \mathcal{D}A^a$ and the $SU(2)$ triplets of scalars $\phi_a{}^i{}_j$. All other fields are projected from the bulk onto the orbifold plane, and there are algebraic equations relating them to the 11-dimensional fields in $\mathcal{S}_{11}$. As discussed above, these relations are trivial for the metric $g_{\mu\nu}$ and the three-form $C_{\mu\nu\rho}$, whilst the scalar τ is given by

$$\tau = \frac{1}{2}\ln\det g_{AB}\,, \tag{5.20}$$

and can be interpreted as an overall scale factor of the orbifold $\mathbb{C}^2/\mathbb{Z}_N$. For the remaining fields, the "gravi-photons" $F^I_{\mu\nu}$ and the "orbifold moduli" $\ell_I{}^{\underline{J}}$, we have to distinguish between the generic $\mathbb{Z}_N$ orbifold with $N > 2$ and the $\mathbb{Z}_2$ orbifold. For $\mathbb{Z}_N$ with $N > 2$ we have four $U(1)$ gauge fields, so that $I = 1, \dots, 4$, and $\ell_I{}^{\underline{J}}$ parameterizes the coset $SO(3,1)/SO(3)$. They are identified with 11-dimensional fields through

$$\begin{aligned} F^I_{\mu\nu} &= -\frac{i}{2}\mathrm{tr}\left(\sigma^I G_{\mu\nu}\right), && (5.21)\\ \ell_I{}^{\underline{J}} &= \frac{1}{2}\mathrm{tr}\left(\bar{\sigma}_I v\sigma^{\underline{J}}v^\dagger\right), && (5.22) \end{aligned}$$

where $G_{\mu\nu} \equiv (G_{\mu\nu p\bar{q}})$, $v \equiv (e^{\tau/4}e^{\bar{p}}{}_{\underline{\bar{q}}})$ and σ^I are the $SO(3,1)$ Pauli matrices as given in Appendix B. For the $\mathbb{Z}_2$ case, we have six $U(1)$ vector fields, so that $I = 1, \dots, 6$, and $\ell_I{}^{\underline{J}}$ parameterizes the coset $SO(3,3)/SO(3)^2$. The field identifications now read

$$\begin{aligned} F^I_{\mu\nu} &= -\frac{1}{4}\mathrm{tr}\left(T^I G_{\mu\nu}\right), && (5.23)\\ \ell_I{}^{\underline{J}} &= \frac{1}{4}\mathrm{tr}\left(\bar{T}_I v T^{\underline{J}} v^T\right), && (5.24) \end{aligned}$$

where this time $G_{\mu\nu} \equiv (G_{\mu\nu AB})$, $v \equiv (e^{\tau/4} e^A{}_{\underline{B}})$, and T^I are the generators of $SO(4)$, as given in Appendix B.

Let us discuss a few elementary properties of the bosonic action (5.11) on the orbifold plane, starting with the gauge-kinetic functions (5.12)–(5.14). The first observation is, that the gauge-kinetic function for the $SU(N)$ vector fields is trivial (to the order we have calculated), which confirms the result of Ref. [16]. On the other hand, we find non-trivial gauge kinetic terms between the $SU(N)$ vectors and the gravi-photons, as well as between the gravi-photons. We also note the appearance of the Chern-Simons term $C \wedge F^a \wedge F_a$, which has been predicted [11] from anomaly cancellation in configurations which involve additional matter fields on conical singularities, but, in our case, simply follows from the structure of seven-dimensional supergravity without any further assumption. We note that, while there is no seven-dimensional scalar field term which depends only on orbifold moduli, the scalar field kinetic terms in (5.11) constitute a complicated sigma model which mixes the orbifold moduli and the scalars in the $SU(N)$ vector multiplets. A further interesting feature is the presence of the seven-dimensional D-term potential in Eq. (5.11). Introducing the matrices $\phi_a \equiv (\phi_a{}^i{}_j)$ and $D^a \equiv (D^{ai}{}_j)$ this potential can be written as

$$V = \frac{1}{4g_{\rm YM}^2} \mathrm{tr}\left(D^a D_a\right) , \tag{5.25}$$

where

$$D^a = \frac{1}{2} e^{\tau} f^a{}_{bc} [\phi^b, \phi^c] . \tag{5.26}$$

The flat directions, $D^a = 0$, of this potential, which correspond to unbroken supersymmetry as can be seen from Eq. (5.9), can be written as

$$\phi^a = v^a \sigma^3 \tag{5.27}$$

with vacuum expectation values v^a. The v^a correspond to elements in the Lie algebra of $SU(N)$ which can be diagonalised into the Cartan sub-algebra. Generic such diagonal matrices break $SU(N)$ to $U(1)^{N-1}$, while larger unbroken groups are possible for non-generic choices. Looking at the scalar field masses induced from the D-term in such a generic situation, we have one massless scalar for each of the non-Abelian gauge fields which is absorbed as their longitudinal degree of freedom. For each of the $N-1$ unbroken Abelian gauge fields, we have all three associated scalars massless, as must be the case from supersymmetry. This situation corresponds exactly to what happens when the orbifold singularity is blown up. We can, therefore, see that within our supergravity construction blowing-up is encoded by the D-term. Further, the Abelian gauge fields in $SU(N)$ correspond to (a truncated version of) the massless vector fields which arise from zero modes of the M-theory three-form on a blown-up orbifold, while the $3(N-1)$ scalars in the Abelian vector fields correspond to the blow-up moduli.

6 Discussion and outlook

In this paper, we have constructed the effective supergravity action for M-theory on the orbifold $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{R}^{1,6}$, by coupling 11-dimensional supergravity, constrained in accordance with the orbifolding, to $SU(N)$ super-Yang-Mills theory located on the seven-dimensional fixed plane of the orbifold.

We have found that the orbifold-constrained fields of 11-dimensional supergravity, when restricted to the orbifold plane, fill out a seven-dimensional supergravity multiplet plus a single $U(1)$ vector multiplet for $N > 2$ and three $U(1)$ vector multiplets for $N = 2$. The seven-dimensional action on the orbifold plane, which has to be added to 11-dimensional supergravity, couples these bulk degrees of freedom to genuine seven-dimensional states in the $SU(N)$ multiplet. We have obtained this action on the orbifold plane by "up-lifting" information from the known action of $\mathcal{N} = 1$ Einstein-Yang-Mills supergravity and identifying 11- and 7-dimensional degrees of freedom appropriately. The resulting 11-/7-dimensional theory is given as an expansion in the parameter $h = \kappa^{5/9}/g_{\mathrm{YM}}$, where κ is the 11-dimensional Newton constant and g_{YM} is the seven-dimensional $SU(N)$ coupling. The bulk theory appears at zeroth order in h, and we have determined the complete set of leading terms on the orbifold plane which are of order h^2. At order h^4 we encounter a singularity due to a delta function square, similar to what happens in Hořava-Witten theory [3]. As in Ref. [3], we assume that this singularity will be resolved in full M-theory, when the finite thickness of the orbifold plane is taken into account, and that it does not invalidate the results at order h^2.

While we have focused on the A-type orbifolds $\mathbb{C}^2/\mathbb{Z}_N$, we expect our construction to work analogously for the other four-dimensional orbifolds of ADE type. Our result represents the proper starting point for compactifications of M-theory on G_2 spaces with singularities of the type $\mathbb{C}^2/\mathbb{Z}_N \times B$, where B is a three-dimensional manifold. We consider this to be the first step in a programme, aiming at developing an explicit supergravity framework for "phenomenological" compactifications of M-theory on singular G_2 spaces. A further important step would be to couple to our action a four-dimensional $\mathcal{N} = 1$ action, describing the matter fields on conical singularities. The structure of such a coupled supergravity in eleven, seven and four dimensions is currently under investigation.

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Appendix

A Spinor Conventions

In this section, we provide the conventions for gamma matrices and spinors in eleven, seven and four dimensions and the relations between them. This split of eleven dimensions into seven plus four arises naturally from the orbifolds $\mathbb{R}^{1,6} \times \mathbb{C}^2/\mathbb{Z}_N$ which we consider in this paper. We need to work out the appropriate spinor decomposition for this product space and, in particular, write 11-dimensional Majorana spinors as a product of seven-dimensional symplectic Majorana spinors with an appropriate basis of four-dimensional spinors. We denote 11-dimensional coordinates by (x^M), with indices $M, N, \ldots = 0, \ldots, 10$. They are split up as $x^M = (x^\mu, y^A)$ with seven-dimensional coordinates x^μ, where $\mu, \nu, \ldots = 0, \ldots, 6$, on $\mathbb{R}^{1,6}$ and four-dimensional coordinates y^A, where $A, B, \ldots = 7, \ldots, 10$, on $\mathbb{C}^2/\mathbb{Z}_N$.

We begin with gamma matrices and spinors in 11-dimensions. The gamma-matrices, Γ^M, satisfy the standard Clifford algebra

$$\{\Gamma^M, \Gamma^N\} = 2g^{MN}, \tag{A.1}$$

where g^{MN} is the metric on the full space $\mathbb{R}^{1,6} \times \mathbb{C}^2/\mathbb{Z}_N$. We define the Dirac conjugate of an 11-dimensional spinor Ψ to be

$$\bar{\Psi} = i\Psi^\dagger\Gamma^0. \tag{A.2}$$

The 11-dimensional charge conjugate is given by

$$\Psi^C = B^{-1}\Psi^*, \tag{A.3}$$

where the charge conjugation matrix B satisfies [30]

$$B\Gamma^M B^{-1} = \Gamma^{M*}, \quad B^*B = \mathbf{1}_{32}. \tag{A.4}$$

In this work, all spinor fields in 11-dimensions are taken to satisfy the Majorana condition, $\Psi^C = \Psi$, thereby reducing Ψ from 32 complex to 32 real degrees of freedom.

Next, we define the necessary conventions for $SO(1,6)$ gamma matrices and spinors in seven dimensions. The gamma matrices, denoted by Υ^μ, satisfy the algebra

$$\{\Upsilon^\mu, \Upsilon^\nu\} = 2g^{\mu\nu}\ , \tag{A.5}$$

where $g_{\mu\nu}$ is the metric on $\mathbb{R}^{1,6}$. The Dirac conjugate of a general eight complex component spinor ψ is defined by

$$\bar{\psi} = i\psi^\dagger\Upsilon^0\ . \tag{A.6}$$

In seven dimensions, the charge conjugation matrix B_8 has the following properties [30]

$$B_8\Upsilon^\mu B_8^{-1} = \Upsilon^{\mu*}, \quad B_8^*B_8 = -\mathbf{1}_8\ . \tag{A.7}$$

The second of these relations implies that charge conjugation, defined by

$$\psi^c = B_8^{-1}\psi^* \tag{A.8}$$

squares to minus one. Hence, one cannot define seven-dimensional $SO(1,6)$ Majorana spinors. However, the supersymmetry algebra in seven dimensions contains an $SU(2)$ R-symmetry and spinors can be naturally assembled into $SU(2)$ doublets ψ^i, where $i, j, \ldots = 1, 2$. Indices $i, j, \ldots$ can be lowered and raised with the two-dimensional Levi-Civita tensor ϵ_{ij} and ϵ^{ij}, normalized so that $\epsilon^{12} = \epsilon_{21} = 1$. With these conventions a symplectic Majorana condition

$$\psi_i = \epsilon_{ij}B_8^{-1}\psi^{*j}, \tag{A.9}$$

can be imposed on an $SU(2)$ doublet ψ^i of spinors, where we have defined $\psi^{*i} \equiv (\psi_i)^*$. All seven-dimensional spinors in this paper are taken to be such symplectic Majorana spinors. Further, in computations with seven-dimensional spinors, the following identities are frequently useful,

$$\bar{\chi}^i\Upsilon^{\mu_1\cdots\mu_n}\psi^j = (-1)^{n+1}\bar{\psi}^j\Upsilon^{\mu_n\cdots\mu_1}\chi^i, \tag{A.10}$$

$$\bar{\chi}^i\Upsilon^{\mu_1\cdots\mu_n}\psi_i = (-1)^n\bar{\psi}^i\Upsilon^{\mu_n\cdots\mu_1}\chi_i. \tag{A.11}$$

Finally, we need to fix conventions for four-dimensional Euclidean gamma matrices and spinors. Four-dimensional gamma matrices, denoted by γ^A, satisfy

$$\{\gamma^A, \gamma^B\} = 2g^{AB}\,, \tag{A.12}$$

with the metric g_{AB} on $\mathbb{C}^2/\mathbb{Z}_N$. The chirality operator, defined by

$$\gamma = \gamma^{\underline{7}}\gamma^{\underline{8}}\gamma^{\underline{9}}\gamma^{\underline{10}}\,, \tag{A.13}$$

satisfies $\gamma^2 = \mathbf{1}_4$. The four-dimensional charge conjugation matrix B_4 satisfies the properties

$$B_4\gamma^A B_4^{-1} = \gamma^{A*}\,, \qquad B_4^* B_4 = -\mathbf{1}_4\,. \tag{A.14}$$

It will often be more convenient to work with complex coordinates $(z^p, \bar{z}^{\bar{p}})$ on $\mathbb{C}^2/\mathbb{Z}_N$, where $p, q, \ldots = 1, 2$ and $\bar{p}, \bar{q}, \ldots = \bar{1}, \bar{2}$. In these coordinates, the Clifford algebra takes the well-known "harmonic oscillator" form

$$\{\gamma^p, \gamma^q\} = 0\,, \qquad \left\{\gamma^{\bar{p}}, \gamma^{\bar{q}}\right\} = 0\,, \qquad \left\{\gamma^p, \gamma^{\bar{q}}\right\} = 2g^{p\bar{q}}\,, \tag{A.15}$$

with creation and annihilation "operators" γ^p and $\gamma^{\bar{p}}$, respectively. In this new basis, complex conjugation of gamma matrices (A.14) is described by

$$B_4\gamma^{\bar{p}}B_4^{-1} = \gamma^{p*}\,, \qquad B_4\gamma^p B_4^{-1} = \gamma^{\bar{p}*}\,. \tag{A.16}$$

A basis of spinors can be obtained by starting with the "vacuum state" Ω, which is annihilated by $\gamma^{\bar{p}}$, that is $\gamma^{\bar{p}}\Omega = 0$, and applying creation operators to it. This leads to the three further states

$$\rho^{\underline{p}} = \frac{1}{\sqrt{2}}\gamma^{\underline{p}}\Omega\,, \qquad \bar{\Omega} = \frac{1}{2}\gamma^{\underline{1}}\gamma^{\underline{2}}\Omega\,. \tag{A.17}$$

In terms of the gamma matrices in complex coordinates, the chirality operator γ can be expressed as

$$\gamma = -1 + \gamma^{\underline{\bar{1}}}\gamma^{\underline{1}} + \gamma^{\underline{\bar{2}}}\gamma^{\underline{2}} - \gamma^{\underline{\bar{1}}}\gamma^{\underline{1}}\gamma^{\underline{\bar{2}}}\gamma^{\underline{2}}\,. \tag{A.18}$$

Hence, the basis $(\Omega, \rho^{\underline{p}}, \bar{\Omega})$ consists of chirality eigenstates satisfying

$$\gamma\Omega = -\Omega\,, \qquad \gamma\bar{\Omega} = -\bar{\Omega}\,, \qquad \gamma\rho^{\underline{p}} = \rho^{\underline{p}}\,. \tag{A.19}$$

For ease of notation, we will write the left-handed states as $(\rho^i) = (\rho^{\underline{1}}, \rho^{\underline{2}})$, where $i, j, \ldots = 1, 2$ and the right-handed states as $(\rho^{\bar{\imath}}) = (\Omega, \bar{\Omega})$ where $\bar{\imath}, \bar{\jmath}, \ldots = \bar{1}, \bar{2}$. Note, it follows from Eq. (A.16) that

$$B_4^{-1}\Omega^* = \bar{\Omega}\,, \qquad B_4^{-1}\rho^{\underline{1}*} = \rho^{\underline{2}}\,. \tag{A.20}$$

Hence ρ^i and $\rho^{\bar{\imath}}$ each form a Majorana conjugate pair of spinors with definite chirality.

We should now discuss the four plus seven split of 11-dimensional gamma matrices and spinors. It is easily verified that the matrices

$$\Gamma^\mu = \Upsilon^\mu \otimes \gamma\,, \qquad \Gamma^A = \mathbf{1}_8 \otimes \gamma^A, \tag{A.21}$$

satisfy the Clifford algebra (A.1) and, hence, constitute a valid set of 11-dimensional gamma-matrices. Further, it is clear that an 11-dimensional charge conjugation matrix B can be obtained from its seven- and four-dimensional counterparts B_8 and B_4 by

$$B = B_8 \otimes B_4 \; . \tag{A.22}$$

A general 11-dimensional Dirac spinor Ψ can now be expanded in terms of the basis $(\rho^i, \rho^{\bar{\imath}})$ of four-dimensional spinors as

$$\Psi = \psi_i(x,y) \otimes \rho^i + \psi_{\bar{\jmath}}(x,y) \otimes \rho^{\bar{\jmath}} \; , \tag{A.23}$$

where ψ_i and $\psi_{\bar{\jmath}}$ are four independent seven-dimensional Dirac spinors. Given the properties of the four-dimensional spinor basis under charge conjugation, a Majorana condition on the 11-dimensional spinor Ψ simply translates into ψ_i and $\psi_{\bar{\jmath}}$ each being symplectic $SO(1,6)$ Majorana spinors.

B Some group-theoretical properties

In this section we summarize some group-theoretical properties related to the coset spaces $SO(3,n)/SO(3) \times SO(n)$ of seven-dimensional EYM supergravity. We focus on the parameterization of these coset spaces in terms of 11-dimensional metric components, which is an essential ingredient in re-writing 11-dimensional supergravity, truncated on the orbifold, into standard seven-dimensional EYM supergravity language.

We begin with the generic $\mathbb{C}^2/\mathbb{Z}_N$ orbifold, where $N > 2$ and $n = 1$, so the relevant coset space is $SO(3,1)/SO(3)$. In this case, it is convenient to use complex coordinates $(z^p, \bar{z}^{\bar{p}})$, where $p,q,\ldots = 1,2$ and $\bar{p},\bar{q},\ldots = \bar{1},\bar{2}$, on the orbifold. After truncating the 11-dimensional metric to be independent of the orbifold coordinates, the surviving degrees of freedom of the orbifold part of the metric can be described by the components $e_p{}^{\underline{p}}$ of the vierbein, see Eqs. (2.7)–(2.16). Extracting the overall scale factor from this, we have a determinant one object $v_p{}^{\underline{p}}$, together with identifications by $SU(2)$ gauge transformations acting on the tangent space index. Hence, $v_p{}^{\underline{p}}$ should be thought of as parameterizing the coset $SL(2,\mathbb{C})/SU(2)$. This space is indeed isomorphic to $SO(3,1)/SO(3)$. To work this out explicitly, it is useful to introduce the map f defined by

$$f(u) = u_I \sigma^I \tag{B.1}$$

which maps four-vectors u_I, where $I,J,\ldots = 1,\ldots,4$, into hermitian matrices $f(u)$. Here the matrices σ^I and their conjugates $\bar{\sigma}^I$ are given by

$$(\sigma^I) = (\sigma^u, \mathbf{1}_2) \; , \qquad (\bar{\sigma}^I) = (-\sigma^u, \mathbf{1}_2) \; , \tag{B.2}$$

where the σ^u, $u = 1,2,3$, are the standard Pauli matrices. They satisfy the following useful identities

$$\mathrm{tr}\left(\sigma^I \bar{\sigma}^J\right) = 2\eta^{IJ}, \tag{B.3}$$

$$\mathrm{tr}\left(\bar{\sigma}^I \sigma^{(J} \bar{\sigma}^{|K|} \sigma^{L)}\right) = 2\left(\eta^{IJ}\eta^{KL} + \eta^{IL}\eta^{JK} - \eta^{IK}\eta^{JL}\right) \; , \tag{B.4}$$

where $I,J,\ldots$ indices are raised and lowered with the Minkowski metric $(\eta_{IJ}) = \mathrm{diag}(-1,-1,-1,+1)$. A key property of the map f is that

$$u_I u^I = \det(f(u)) \tag{B.5}$$

for four-vectors u_I. This property is crucial in demonstrating that the map F defined by

$$F(v)u = f^{-1}\left(vf(u)v^\dagger\right) \tag{B.6}$$

is a group homeomorphism $F : SL(2,\mathbb{C}) \to SO(3,1)$. Solving explicitly for the $SO(3,1)$ images $\ell_I{}^{\underline{J}} = (F(v))_I{}^{\underline{J}}$ one finds

$$\ell_I{}^{\underline{J}} = \frac{1}{2}\mathrm{tr}\left(\bar{\sigma}_I v \sigma^J v^\dagger\right) \ . \tag{B.7}$$

This map induces the desired map $SL(2,\mathbb{C})/SU(2) \to SO(3,1)/SO(3)$ between the cosets.

The structure is analogous, although slightly more involved, for the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, where $n = 3$ and the relevant coset space is $SO(3,3)/SO(3)^2$. In this case, it is more appropriate to work with real coordinates y^A on the orbifold, where $A, B, \ldots = 7, 8, 9, 10$. The orbifold part of the truncated 11-dimensional metric, rescaled to determinant one, is then described by the vierbein $v_A{}^{\underline{A}}$ in real coordinates, which parameterizes the coset $SL(4,\mathbb{R})/SO(4)$. The map f now identifies $SO(3,3)$ vectors u with elements of the $SO(4)$ Lie algebra according to

$$f(u) = u_I T^I \ , \tag{B.8}$$

where T^I, with $I, J, \ldots = 1, \ldots, 6$ is a basis of anti-symmetric 4×4 matrices. We would like to choose these matrices so that the first four, $T^1, \ldots, T^4$ correspond to the Pauli matrices $\sigma^1, \ldots, \sigma^4$ of the previous $N > 2$ case, when written in real coordinates. This ensures that our result for $N = 2$ indeed exactly reduces to the one for $N > 2$ when the additional degrees of freedom are "switched off" and, hence, the action for both cases can be written in a uniform language. It turns out that such a choice of matrices is given by

$$T^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{B.9}$$

$$T^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad T^4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{B.10}$$

The two remaining matrices can be taken as

$$T^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad T^6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{B.11}$$

Note that $T^{1,2,3}$ and $T^{4,5,6}$ form the two sets of $SU(2)$ generators within the $SO(4)$ Lie algebra. We may introduce a "dual" to the six T^I matrices, analogous to the definition of the $\bar{\sigma}^I$ matrices of the $N > 2$ case, which will prove useful in many calculations. We define

$$\left(\overline{T}^I\right)^{AB} = -\frac{1}{2}\left(T^I\right)_{CD}\epsilon^{ABCD} \tag{B.12}$$

which has the simple form

$$(\overline{T}^I) = (T^u, -T^\alpha), \tag{B.13}$$

where $u, v, \ldots = 1, 2, 3$ and $\alpha, \beta, \ldots = 4, 5, 6$. Indices $I, J, \ldots$ are raised and lowered with the metric $(\eta_{IJ}) = \mathrm{diag}(-1, -1, -1, +1, +1, +1)$. The matrices T^I satisfy the following useful identities

$$\mathrm{tr}\left(T^I\, \overline{T}^J\right) = 4\eta^{IJ}, \tag{B.14}$$

$$\left(T^I\right)_{AB}\left(T^J\right)_{CD}\eta_{IJ} = 2\epsilon_{ABCD}, \tag{B.15}$$

$$\left(T^u\right)_{AB}\left(T^v\right)_{CD}\delta_{uv} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} - \epsilon_{ABCD}, \tag{B.16}$$

$$(T^\alpha)_{AB}(T^\beta)_{CD}\delta_{\alpha\beta} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + \epsilon_{ABCD}. \tag{B.17}$$

Key property of the map f is

$$(u_I u^I)^2 = \det(f(u)) \tag{B.18}$$

for any $SO(3,3)$ vector u_I. This property can be used to show that the map F defined by

$$F(v)u = f^{-1}\left(vf(u)v^T\right) \tag{B.19}$$

is a group homeomorphism $F : SL(4,\mathbb{R}) \to SO(3,3)$. Solving for the $SO(3,3)$ images $\ell_I{}^{\underline{J}} = (F(v))_I{}^{\underline{J}}$ one finds

$$\ell_I{}^{\underline{J}} = \frac{1}{4}\mathrm{tr}\left(\bar{T}_I v T^J v^T\right) . \tag{B.20}$$

This induces the desired map between the cosets $SL(4,\mathbb{R})/SO(4)$ and $SO(3,3)/SO(3)^2$.

C Einstein-Yang-Mills supergravity in seven dimensions

In this section we will give a self-contained summary of minimal, $\mathcal{N} = 1$ Einstein-Yang-Mills (EYM) supergravity in seven dimensions. Although the theory may be formulated in two equivalent ways, here we treat only the version in which the gravity multiplet contains a three-form potential $C_{\mu\nu\rho}$ [27], rather than the dual formulation in terms of a two-index antisymmetric field $B_{\mu\nu}$ which has been studied in Refs. [25, 31]. This three-form formulation is better suited for our application to M-theory. The theory has an $SU(2)$ rigid R-symmmetry that may be gauged and the resulting massive theories were first constructed in Refs. [26, 32, 33]. The seven-dimensional supergravities we obtain by truncating M-theory are not massive and, for this reason, we will not consider such theories with gauged R-symmetry. The seven-dimensional pure supergravity theory can also be coupled to M vector multiplets [27, 34, 35, 36], transforming under a Lie group $G = U(1)^n \times H$, where H is semi-simple, in which case the vector multiplet scalars parameterize the coset space $SO(3,M)/SO(3) \times SO(M)$. In this Appendix, we will first review seven-dimensional $\mathcal{N} = 1$ EYM supergravity with such a gauge group G. This theory is used in the main part of the paper to construct the complete action for low-energy M-theory on the orbifolds $\mathbb{R}^{1,6} \times \mathbb{C}^2/\mathbb{Z}_N$. The truncation of M-theory on these orbifolds to seven dimensions leads to a $d = 7$ EYM supergravity with gauge group $U(1)^n \times SU(N)$, where $n = 1$ for $N > 2$ and $n = 3$ for $N = 2$. Here, the $U(1)^n$ part of the gauge group originates from truncated bulk states, while the $SU(N)$ non-Abelian part corresponds to the additional states which arise on the orbifold fixed plane. Since we are constructing the coupled

11-/7-dimensional theory as an expansion in $SU(N)$ fields, the crucial building block is a version of $d = 7$ EYM supergravity with gauge group $U(1)^n \times SU(N)$, expanded around the supergravity and $U(1)^n$ part. This expanded version of the theory is presented in the final part of this Appendix.

C.1 General action and supersymmetry transformations

The field content of gauged $d = 7$, $\mathcal{N} = 1$ EYM supergravity consists of two types of multiplets. The first, the gravitational multiplet, contains a graviton $g_{\mu\nu}$ with associated vielbein $e_\mu{}^{\underline{\nu}}$, a gravitino ψ^i_μ, a symplectic Majorana spinor χ^i, an $SU(2)$ triplet of Abelian vector fields $A_\mu{}^i{}_j$ with field strengths $F^i{}_j = \mathrm{d}A^i{}_j$, a three form field $C_{\mu\nu\rho}$ with field strength $G = \mathrm{d}C$, and a real scalar σ. So, in summary we have

$$\left(g_{\mu\nu}, C_{\mu\nu\rho}, A_\mu{}^i{}_j, \sigma, \psi^i_\mu, \chi^i\right) \ . \tag{C.1}$$

Here, $i, j, \ldots = 1, 2$ are $SU(2)$ R-symmetry indices. The second type is the vector multiplet, which contains gauge vectors A^a_μ with field strengths $F^a = \mathcal{D}A^a$, gauginos λ^{ai} and $SU(2)$ triplets of real scalars $\phi^{ai}{}_j$. In summary, we have

$$\left(A^a_\mu, \phi^{ai}{}_j, \lambda^{ai}\right) \ , \tag{C.2}$$

where $a, b, \ldots = 4, \ldots, M + 3$ are Lie algebra indices of the gauge group G.

It is sometimes useful to combine all vector fields, the three Abelian ones in the gravity multiplet as well as the ones in the vector multiplets, into a single $SO(3, M)$ vector

$$(A^{\tilde{I}}_\mu) = \left(A_\mu{}^i{}_j, A^a_\mu\right) \ , \tag{C.3}$$

where $\tilde{I}, \tilde{J}, \ldots = 1, \ldots, M + 3$. Under this combination, the corresponding field strengths are given by

$$F^{\tilde{I}}_{\mu\nu} = 2\partial_{[\mu}A^{\tilde{I}}_{\nu]} + f_{\tilde{J}\tilde{K}}{}^{\tilde{I}} A^{\tilde{J}}_\mu A^{\tilde{K}}_\nu , \tag{C.4}$$

where $f_{bc}{}^a$ are the structure constants for G and all other components of $f_{\tilde{J}\tilde{K}}{}^{\tilde{I}}$ vanish.

The coset space $SO(3, M)/SO(3) \times SO(M)$ is described by a $(3 + M) \times (3 + M)$ matrix $L_{\tilde{I}}{}^{\underline{\tilde{J}}}$, which depends on the $3M$ vector multiplet scalars and satisfies the $SO(3, M)$ orthogonality condition

$$L_{\tilde{I}}{}^{\underline{\tilde{J}}} L_{\tilde{K}}{}^{\underline{\tilde{L}}} \eta_{\underline{\tilde{J}}\underline{\tilde{L}}} = \eta_{\tilde{I}\tilde{K}} \tag{C.5}$$

with $(\eta_{\tilde{I}\tilde{J}}) = (\eta_{\underline{\tilde{I}}\underline{\tilde{J}}}) = \mathrm{diag}(-1, -1, -1, +1, \ldots, +1)$. Here, indices $\tilde{I}, \tilde{J}, \ldots = 1, \ldots, (M+3)$ transform under $SO(3, M)$. Their flat counterparts $\underline{\tilde{I}}, \underline{\tilde{J}}, \ldots$ decompose into a triplet of $SU(2)$, corresponding to the gravitational directions and M remaining directions corresponding to the vector multiplets. Thus we can write $L_{\tilde{I}}{}^{\underline{\tilde{J}}} \to \left(L_{\tilde{I}}{}^u, L_{\tilde{I}}{}^a\right)$, where $u = 1, 2, 3$. The adjoint $SU(2)$ index u can be converted into a pair of fundamental $SU(2)$ indices by multiplication with the Pauli matrices, that is,

$$L_{\tilde{I}}{}^i{}_j = \frac{1}{\sqrt{2}} L_{\tilde{I}}{}^u \left(\sigma_u\right)^i{}_j \ . \tag{C.6}$$

There are obviously many ways in which one can parameterize the coset space $SO(3,M)/SO(3) \times SO(M)$ in terms of the physical vector multiplet scalar degrees of freedom ${\phi_a}^i{}_j$. A simple parameterization of this coset in terms of $\Phi \equiv ({\phi_a}^u)$ is given by

$$L_{\tilde{I}}{}^{\underline{\tilde{J}}} = \left(\exp \begin{bmatrix} 0 & \Phi^T \\ \Phi & 0 \end{bmatrix} \right)_{\tilde{I}}{}^{\underline{\tilde{J}}}. \tag{C.7}$$

In the final paragraph of this appendix, when we expand seven-dimensional supergravity, we will use a different parameterization, which is better adapted to this task. The Maurer-Cartan form of the matrix L, defined by $L^{-1}\mathcal{D}L$, is needed to write down the theory. The components P and Q are given explicitly by

$$P_{\mu a}{}^i{}_j = L^{\tilde{I}}{}_a \left(\delta^{\tilde{K}}_{\tilde{I}} \partial_\mu + f_{\tilde{I}\tilde{J}}{}^{\tilde{K}} A^{\tilde{J}}_\mu \right) L_{\tilde{K}}{}^i{}_j, \tag{C.8}$$

$$Q_\mu{}^i{}_j = L^{\tilde{I}i}{}_k \left(\delta^{\tilde{K}}_{\tilde{I}} \partial_\mu + f_{\tilde{I}\tilde{J}}{}^{\tilde{K}} A^{\tilde{J}}_\mu \right) L_{\tilde{K}}{}^k{}_j. \tag{C.9}$$

The final ingredients needed are the following projections of the structure constants

$$\begin{aligned} D &= i f_{ab}{}^c L^{ai}{}_k L^{bj}{}_i L_c{}^k{}_j, \\ D^{ai}{}_j &= i f_{bc}{}^d L^{bi}{}_k L^{ck}{}_j L_d{}^a, \\ D_{ab}{}^i{}_j &= f_{cd}{}^e L_a{}^c L_b{}^d L_e{}^i{}_j. \end{aligned} \tag{C.10}$$

It is worth mentioning that invariance of the theory under the gauge group G and the R-symmetry group $SU(2)$ requires that the Maurer-Cartan forms P and Q transform covariantly. It can be shown that this is the case, if and only if the "extended" set of structure constants $f_{\tilde{I}\tilde{J}}{}^{\tilde{K}}$ satisfy the condition

$$f_{\tilde{I}\tilde{J}}{}^{\tilde{L}} \eta_{\tilde{L}\tilde{K}} = f_{[\tilde{I}\tilde{J}}{}^{\tilde{L}} \eta_{\tilde{K}]\tilde{L}}. \tag{C.11}$$

For any direct product factor of the total gauge group, this condition can be satisfied in two ways. Either, the structure constants are trivial, or the metric $\eta_{\tilde{I}\tilde{J}}$ is the Cartan-Killing metric of this factor. In our particular case, the condition (C.11) is satisfied by making use of both these possibilities. The structure constants vanish for the "gravitational" part of the gauge group and the $U(1)^n$ part within G. For the semi-simple part H of G, one can always choose a basis, so its Cartan-Killing metric is simply the Kronecker delta.

With everything in place, we now write down the Lagrangian for the theory. Setting coupling

constants to one, and neglecting four-fermi terms, it is given by [27]

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm YM} = & \frac{1}{2}R - \frac{1}{2}\bar\psi^i_\mu \Upsilon^{\mu\nu\rho}\hat{\mathcal{D}}_\nu\psi_{\rho i} - \frac{1}{96}e^{4\sigma}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} - \frac{1}{2}\bar\chi^i\Upsilon^\mu\hat{\mathcal{D}}_\mu\chi_i - \frac{5}{2}\partial_\mu\sigma\partial^\mu\sigma \\
& + \frac{\sqrt{5}}{2}\left(\bar\chi^i\Upsilon^{\mu\nu}\psi_{\mu i} + \bar\chi^i\psi^\nu_i\right)\partial_\nu\sigma + e^{2\sigma}G_{\mu\nu\rho\sigma}\Bigg[\frac{1}{192}\left(\bar\psi^i_\lambda\Upsilon^{\lambda\mu\nu\rho\sigma\tau}\psi_{\tau i} + 12\bar\psi^{\mu i}\Upsilon^{\nu\rho}\psi^\sigma_i\right) \\
& \qquad + \frac{1}{48\sqrt{5}}\left(4\bar\chi^i\Upsilon^{\mu\nu\rho}\psi^\sigma_i - \bar\chi^i\Upsilon^{\mu\nu\rho\sigma\tau}\psi_{\tau i}\right) - \frac{1}{320}\bar\chi^i\Upsilon^{\mu\nu\rho\sigma}\chi_i\Bigg] \\
& - \frac{1}{4}e^{-2\sigma}\left(L_{\tilde I}{}^i{}_j L_{\tilde J}{}^j{}_i + L^a_{\tilde I}L_{\tilde J a}\right)F^{\tilde I}_{\mu\nu}F^{\tilde J\mu\nu} - \frac{1}{2}\bar\lambda^{ai}\Upsilon^\mu\hat{\mathcal{D}}_\mu\lambda_{ai} - \frac{1}{2}P_\mu{}^{ai}{}_j P^\mu{}_a{}^j{}_i \\
& - \frac{1}{\sqrt{2}}\left(\bar\lambda^{ai}\Upsilon^{\mu\nu}\psi_{\mu j} + \bar\lambda^{ai}\psi^\nu_j\right)P_{\nu a}{}^j{}_i + \frac{1}{192}e^{2\sigma}G_{\mu\nu\rho\sigma}\bar\lambda^{ai}\Upsilon^{\mu\nu\rho\sigma}\lambda_{ai} \\
& - ie^{-\sigma}F^{\tilde I}_{\mu\nu}L_{\tilde I}{}^j{}_i\Bigg[\frac{1}{4\sqrt{2}}\left(\bar\psi^i_\rho\Upsilon^{\mu\nu\rho\sigma}\psi_{\sigma j} + 2\bar\psi^{\mu i}\psi^\nu_j\right) + \frac{3}{20\sqrt{2}}\bar\chi^i\Upsilon^{\mu\nu}\chi_j \\
& \qquad - \frac{1}{4\sqrt{2}}\bar\lambda^{ai}\Upsilon^{\mu\nu}\lambda_{aj} + \frac{1}{2\sqrt{10}}\left(\bar\chi^i\Upsilon^{\mu\nu\rho}\psi_{\rho j} - 2\bar\chi^i\Upsilon^\mu\psi^\nu_j\right)\Bigg] \\
& + e^{-\sigma}F^{\tilde I}_{\mu\nu}L_{\tilde I a}\left[\frac{1}{4}\left(2\bar\lambda^{ai}\Upsilon^\mu\psi^\nu_i - \bar\lambda^{ai}\Upsilon^{\mu\nu\rho}\psi_{\rho i}\right) + \frac{1}{2\sqrt{5}}\bar\lambda^{ai}\Upsilon^{\mu\nu}\chi_i\right] \\
& + \frac{5}{180}e^{2\sigma}\left(D^2 - 9D^{ai}{}_j D_a{}^j{}_i\right) - \frac{i}{\sqrt{2}}e^\sigma D_{ab}{}^i{}_j\bar\lambda^{aj}\lambda^b_i + \frac{i}{2}e^\sigma D_a{}^i{}_j\left(\bar\psi^j_\mu\Upsilon^\mu\lambda^a_i + \frac{2}{\sqrt{5}}\bar\chi^j\lambda_i{}^a\right) \\
& + \frac{1}{60\sqrt{2}}e^\sigma D\left(5\bar\psi^i_\mu\Upsilon^{\mu\nu}\psi_{\nu i} + 2\sqrt{5}\bar\psi^i_\mu\Upsilon^\mu\chi_i + 3\bar\chi^i\chi_i - 5\bar\lambda^{ai}\lambda_{ai}\right) \\
& - \frac{1}{96}\epsilon^{\mu\nu\rho\sigma\kappa\lambda\tau}C_{\mu\nu\rho}F^{\tilde I}_{\sigma\kappa}F_{\tilde I\lambda\tau}. \qquad \text{(C.12)}
\end{aligned}
$$

The covariant derivatives that appear here are given by

$$
\hat{\mathcal{D}}_\mu\psi_{\nu i} = \partial_\mu\psi_{\nu i} + \frac{1}{2}Q_{\mu i}{}^j\psi_{\nu j} - \Gamma^\rho_{\mu\nu}\psi_{\rho i} + \frac{1}{4}\omega_\mu{}^{\underline{\mu\nu}}\Upsilon_{\underline{\mu\nu}}\psi_{\nu i}, \qquad \text{(C.13)}
$$

$$
\hat{\mathcal{D}}_\mu\chi_i = \partial_\mu\chi_i + \frac{1}{2}Q_{\mu i}{}^j\chi_j + \frac{1}{4}\omega_\mu{}^{\underline{\mu\nu}}\Upsilon_{\underline{\mu\nu}}\chi_i, \qquad \text{(C.14)}
$$

$$
\hat{\mathcal{D}}_\mu\lambda_{ai} = \partial_\mu\lambda_{ai} + \frac{1}{2}Q_{\mu i}{}^j\lambda_{aj} + \frac{1}{4}\omega_\mu{}^{\underline{\mu\nu}}\Upsilon_{\underline{\mu\nu}}\lambda_{ai} + f_{ab}{}^c A^b_\mu\lambda_{ci}. \qquad \text{(C.15)}
$$

The associated supersymmetry transformations, parameterized by the spinor ε_i, are, up to cubic fermion terms, given by

$$
\delta\sigma = \frac{1}{\sqrt{5}}\bar\chi^i\varepsilon_i \ ,
$$

$$
\delta e_\mu{}^{\underline{\nu}} = \bar\varepsilon^i\Upsilon^{\underline{\nu}}\psi_{\mu i} \ ,
$$

$$
\begin{aligned}
\delta\psi_{\mu i} &= 2\hat{\mathcal{D}}_\mu\varepsilon_i - \frac{1}{80}\left(\Upsilon_\mu{}^{\nu\rho\sigma\eta} - \frac{8}{3}\delta_\mu^\nu\Upsilon^{\rho\sigma\eta}\right)\varepsilon_i G_{\nu\rho\sigma\eta}e^{2\sigma} \\
&\quad + \frac{i}{5\sqrt{2}}\left(\Upsilon_\mu{}^{\nu\rho} - 8\delta_\mu^\nu\Upsilon^\rho\right)\varepsilon_j F_{\nu\rho}^{\tilde{I}} L_{\tilde{I}}{}^j{}_i e^{-\sigma} - \frac{1}{15\sqrt{2}}e^\sigma\Upsilon_\mu\varepsilon_i D\ , \\
\delta\chi_i &= \sqrt{5}\Upsilon^\mu\varepsilon_i\partial_\mu\sigma - \frac{1}{24\sqrt{5}}\Upsilon^{\mu\nu\rho\sigma}\varepsilon_i G_{\mu\nu\rho\sigma}e^{2\sigma} \ - \frac{i}{\sqrt{10}}\Upsilon^{\mu\nu}\varepsilon_j F_{\mu\nu}^{\tilde{I}} L_{\tilde{I}}{}^j{}_i e^{-\sigma} + \frac{1}{3\sqrt{10}}e^\sigma\varepsilon_i D\ , \\
\delta C_{\mu\nu\rho} &= \left(-3\bar{\psi}^i_{[\mu}\Upsilon_{\nu\rho]}\varepsilon_i - \frac{2}{\sqrt{5}}\bar{\chi}^i\Upsilon_{\mu\nu\rho}\varepsilon_i\right)e^{-2\sigma}\ , \qquad\qquad \text{(C.16)} \\
L_{\tilde{I}}{}^i{}_j\delta A_\mu^{\tilde{I}} &= \left[i\sqrt{2}\left(\bar{\psi}^i_\mu\varepsilon_j - \frac{1}{2}\delta^i_j\bar{\psi}^k_\mu\varepsilon_k\right) - \frac{2i}{\sqrt{10}}\left(\bar{\chi}^i\Upsilon_\mu\varepsilon_j - \frac{1}{2}\delta^i_j\bar{\chi}^k\Upsilon_\mu\varepsilon_k\right)\right]e^\sigma\ , \\
L_{\tilde{I}}{}^a\delta A_\mu^{\tilde{I}} &= \bar{\varepsilon}^i\Upsilon_\mu\lambda^a_i e^\sigma\ , \\
\delta L_{\tilde{I}}{}^i{}_j &= -i\sqrt{2}\bar{\varepsilon}^i\lambda_{aj}L_{\tilde{I}}{}^a + \frac{i}{\sqrt{2}}\bar{\varepsilon}^k\lambda_{ak}L^a_{\tilde{I}}\delta^i_j\ , \\
\delta L_{\tilde{I}}{}^a &= -i\sqrt{2}\bar{\varepsilon}^i\lambda^a_j L_{\tilde{I}}{}^j{}_i\ , \\
\delta\lambda^a_i &= -\frac{1}{2}\Upsilon^{\mu\nu}\varepsilon_i F_{\mu\nu}^{\tilde{I}} L^a_{\tilde{I}} e^{-\sigma} + \sqrt{2}i\Upsilon^\mu\varepsilon_j P_\mu{}^{aj}{}_i \ - e^\sigma\varepsilon_j D^{aj}{}_i\ .
\end{aligned}
$$

C.2 A perturbative expansion

In this final section we expand the EYM supergravity of Section C.1 around its supergravity and $U(1)^n$ part. The parameter for the expansion is $h := \kappa_7/g_{\rm YM}$, where κ_7 is the coupling for gravity and $U(1)^n$ and $g_{\rm YM}$ is the coupling for H, the non-Abelian part of the gauge group. To determine the order in h of each term in the Lagrangian, we need to fix a convention for the energy dimensions of the fields. Within the gravity and $U(1)$ vector multiplets, we assign energy dimension 0 to bosonic fields and energy dimension 1/2 to fermionic fields. For the H vector multiplet, we assign energy dimension 1 to the bosons and 3/2 to the fermions. With these conventions we can write

$$
\mathcal{L}_{\rm YM} = \kappa_7^{-2}\left(\mathcal{L}_{(0)} + h^2\mathcal{L}_{(2)} + h^4\mathcal{L}_{(4)} + \ldots\right), \qquad \text{(C.17)}
$$

where the $\mathcal{L}_{(m)}$, $m = 0, 2, 4, \ldots$ are independent of h. The first term in this series is the Lagrangian for EYM supergravity with gauge group $U(1)^n$, whilst the second term contains the leading order non-Abelian gauge multiplet terms. We will write down these first two terms and provide truncated supersymmetry transformation laws suitable for the theory at this order.

In order to carry out the expansion, it is necessary to cast the field content in a form where the H vector multiplet fields and the gravity/$U(1)^n$ vector multiplet fields are disentangled. To this end, we decompose the single Lie algebra indices $a, b, \ldots = 4, \ldots, M+3$ used in Section C.1 into indices $\alpha, \beta, \ldots = 4, \ldots, 3+n$ that label the $U(1)$ directions and redefined indices $a, b, \ldots = n+4, \ldots, M+3$ that are Lie algebra indices of H. This makes the disentanglement straightforward for most of the fields. For example, vector fields, which naturally combine into the single entity $A_\mu^{\tilde{I}}$, can simply be decomposed as $A_\mu^{\tilde{I}} = (A_\mu^I, A_\mu^a)$, where A_μ^I, $I = 1, \ldots, n+3$, refers to the three vector fields in the gravity multiplet and the $U(1)^n$ vector fields, and A_μ^a denotes the H vector fields. Similarly, the $U(1)$ gauginos are denoted by $\lambda_{\alpha i}$, whilst the H gauginos are denoted by λ_{ai}. The situation is somewhat more complicated for the vector multiplet scalar fields, which, as discussed, all together combine into

the single coset $SO(3,M)/SO(3)\times SO(M)$, parameterized by the $SO(3,M)$ matrix L. It is necessary to find an explicit form for L, which separates the $3n$ scalars in the $U(1)^n$ vector multiplets from the $3(M-n)$ scalars in the H vector multiplet. To this end, we note that, in the absence of the H states, the $U(1)^n$ states parameterize a $SO(3,n)/SO(3)\times SO(n)$ coset, described by $(3+n)\times(3+n)$ matrices $\ell_I{}^{\underline{I}} = (\ell_I{}^u, \ell_I{}^\alpha)$. Here, $\ell \equiv (\ell_I{}^u)$ are $(3+n)\times 3$ matrices where the index $u = 1,2,3$ corresponds to the three "gravity" directions and $m \equiv (\ell_I{}^\alpha)$ are $(3+n)\times n$ matrices with $\alpha = 4,\ldots,n+3$ labeling the $U(1)^n$ directions. Let us further denote the $SU(N)$ scalars by $\Phi \equiv (\phi_a{}^u)$. Then we can construct approximate representatives L of the large coset $SO(3,M)/SO(3)\times SO(M)$ by expanding, to the appropriate order in Φ, around the small coset $SO(3,n)/SO(3)\times SO(n)$ represented by ℓ and m. Neglecting terms of cubic and higher order in Φ, this leads to

$$L = \begin{pmatrix} \ell + \frac{1}{2}h^2\ell\Phi^T\Phi & m & h\ell\Phi^T \\ h\Phi & 0 & \mathbf{1}_{M-n} + \frac{1}{2}h^2\Phi\Phi^T \end{pmatrix} . \tag{C.18}$$

We note that the neglected Φ terms are of order h^3 and higher and, since we are aiming to construct the action only up to terms of order h^2, are, therefore, not relevant in the present context.

For the expansion of the action it is useful to re-write the coset parameterization (C.18) and the associated Maurer-Cartan forms P and Q in component form. We find

$$L_I{}^i{}_j = \ell_I{}^i{}_j + \frac{1}{2}h^2\ell_I{}^k{}_l\phi^{al}{}_k\phi_a{}^i{}_j \ , \tag{C.19}$$

$$L_I{}^\alpha = h\ell_I{}^\alpha, \tag{C.20}$$

$$L_I{}^a = h\ell_I{}^i{}_j\phi^{aj}{}_i, \tag{C.21}$$

$$L_a{}^i{}_j = h\phi_a{}^i{}_j \ , \tag{C.22}$$

$$L_a{}^\alpha = 0 \ , \tag{C.23}$$

$$L_a{}^b = \delta_a{}^b + \frac{1}{2}h^2\phi_a{}^i{}_j\phi^{bj}{}_i \ , \tag{C.24}$$

$$P_{\mu\alpha}{}^i{}_j = p_{\mu\alpha}{}^i{}_j + \frac{1}{2}h^2 p_{\mu\alpha}{}^k{}_l\phi^{al}{}_k\phi_a{}^i{}_j \ , \tag{C.25}$$

$$P_{\mu a}{}^i{}_j = -h\hat{\mathcal{D}}_\mu\phi_a{}^i{}_j \ , \tag{C.26}$$

$$Q_\mu{}^i{}_j = q_\mu{}^i{}_j + \frac{1}{2}h^2\left(\phi^{ai}{}_k\hat{\mathcal{D}}_\mu\phi_a{}^k{}_j - \phi_a{}^k{}_j\hat{\mathcal{D}}_\mu\phi^{ai}{}_k\right) \ , \tag{C.27}$$

where p and q are the Maurer-Cartan forms associated with the small coset matrix ℓ. Thus

$$p_{\mu\alpha}{}^i{}_j = \ell^I{}_\alpha\partial_\mu\ell_I{}^i{}_j, \tag{C.28}$$

$$q_\mu{}^i{}_j{}^k{}_l = \ell^{Ii}{}_j\partial_\mu\ell_I{}^k{}_l \ , \tag{C.29}$$

$$q_\mu{}^i{}_j = \ell^{Ii}{}_k\partial_\mu\ell_I{}^k{}_j \ . \tag{C.30}$$

The covariant derivative of the H vector multiplet scalar $\phi_a{}^i{}_j$ is given by

$$\hat{\mathcal{D}}_\mu\phi_a{}^i{}_j = \partial_\mu\phi_a{}^i{}_j - q_\mu{}^i{}_j{}^k{}_l\phi_a{}^l{}_k + f_{ab}{}^c A^b_\mu\phi_c{}^i{}_j. \tag{C.31}$$

Using the expressions above, it is straightforward to perform the expansion of $\mathcal{L}_{\rm YM}$ up to order $h^2 \sim g_{\rm YM}^{-2}$. It is given by

$$
\begin{aligned}
\mathcal{L}_{\rm YM} = \frac{1}{\kappa_7^2}\sqrt{-g}\Big\{ & \frac{1}{2}R - \frac{1}{2}\bar{\psi}^i_\mu \Upsilon^{\mu\nu\rho}\hat{\mathcal{D}}_\nu \psi_{\rho i} - \frac{1}{4}e^{-2\sigma}\left(\ell_I{}^i{}_j \ell_J{}^j{}_i + \ell_I{}^\alpha \ell_{J\alpha}\right) F^I_{\mu\nu}F^{J\mu\nu} \\
& - \frac{1}{96}e^{4\sigma}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} - \frac{1}{2}\bar{\chi}^i\Upsilon^\mu\hat{\mathcal{D}}_\mu\chi_i - \frac{5}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{\sqrt{5}}{2}\left(\bar{\chi}^i\Upsilon^{\mu\nu}\psi_{\mu i} + \bar{\chi}^i\psi^\nu_i\right)\partial_\nu\sigma \\
& - \frac{1}{2}\bar{\lambda}^{\alpha i}\Upsilon^\mu\hat{\mathcal{D}}_\mu\lambda_{\alpha i} - \frac{1}{2}p_{\mu\alpha}{}^i{}_j p^{\mu\alpha j}{}_i - \frac{1}{\sqrt{2}}\left(\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\psi_{\mu j} + \bar{\lambda}^{\alpha i}\psi^\nu_j\right)p_{\nu\alpha}{}^j{}_i \\
& + e^{2\sigma}G_{\mu\nu\rho\sigma}\Big[\frac{1}{192}\left(12\bar{\psi}^{\mu i}\Upsilon^{\nu\rho}\psi^\sigma_i + \bar{\psi}^i_\lambda\Upsilon^{\lambda\mu\nu\rho\sigma\tau}\psi_{\tau i}\right) + \frac{1}{48\sqrt{5}}\left(4\bar{\chi}^i\Upsilon^{\mu\nu\rho}\psi^\sigma_i\right. \\
& \qquad \left. -\bar{\chi}^i\Upsilon^{\mu\nu\rho\sigma\tau}\psi_{\tau i}\right) - \frac{1}{320}\bar{\chi}^i\Upsilon^{\mu\nu\rho\sigma}\chi_i + \frac{1}{192}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu\rho\sigma}\lambda_{\alpha i}\Big] \\
& - ie^{-\sigma}F^I_{\mu\nu}\ell_I{}^j{}_i\Big[\frac{1}{4\sqrt{2}}\left(\bar{\psi}^i_\rho\Upsilon^{\mu\nu\rho\sigma}\psi_{\sigma j} + 2\bar{\psi}^{\mu i}\psi^\nu_j\right) + \frac{1}{2\sqrt{10}}\left(\bar{\chi}^i\Upsilon^{\mu\nu\rho}\psi_{\rho j} - 2\bar{\chi}^i\Upsilon^\mu\psi^\nu_j\right) \\
& \qquad + \frac{3}{20\sqrt{2}}\bar{\chi}^i\Upsilon^{\mu\nu}\chi_j - \frac{1}{4\sqrt{2}}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\lambda_{\alpha j}\Big] \\
& + e^{-\sigma}F^I_{\mu\nu}\ell_{I\alpha}\Big[\frac{1}{4}\left(2\bar{\lambda}^{\alpha i}\Upsilon^\mu\psi^\nu_i - \bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu\rho}\psi_{\rho i}\right) + \frac{1}{2\sqrt{5}}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\chi_i\Big] \\
& - \frac{1}{96}\epsilon^{\mu\nu\rho\sigma\kappa\lambda\tau}C_{\mu\nu\rho}F^I_{\sigma\kappa}F_{I\lambda\tau}\Big\} \\
+ \frac{1}{g^2_{\rm YM}}\sqrt{-g}\Big\{ & -\frac{1}{4}e^{-2\sigma}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{2}\hat{\mathcal{D}}_\mu\phi_a{}^i{}_j\hat{\mathcal{D}}^\mu\phi^{aj}{}_i - \frac{1}{2}\bar{\lambda}^{ai}\Upsilon^\mu\hat{\mathcal{D}}_\mu\lambda_{ai} \\
& - e^{-2\sigma}\ell_I{}^i{}_j\phi_a{}^j{}_i F^I_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}e^{-2\sigma}\ell_I{}^i{}_j\phi_a{}^j{}_i\ell_J{}^k{}_l\phi^{al}{}_k F^I_{\mu\nu}F^J_{\mu\nu} \\
& - \frac{1}{2}p_{\mu\alpha}{}^i{}_j\phi_a{}^j{}_i p^{\mu\alpha k}{}_l\phi^{al}{}_k + \frac{1}{4}\phi_a{}^i{}_k\hat{\mathcal{D}}_\mu\phi^{ak}{}_j\bar{\lambda}^{\alpha j}\Upsilon^\mu\lambda_{\alpha i} \\
& - \frac{1}{\sqrt{2}}\left(\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\psi_{\mu j} + \bar{\lambda}^{\alpha i}\psi^\nu_j\right)\phi_a{}^j{}_i\phi^{ak}{}_l p_{\nu\alpha}{}^l{}_k - \frac{1}{\sqrt{2}}\left(\bar{\lambda}^{ai}\Upsilon^{\mu\nu}\psi_{\mu j} + \bar{\lambda}^{ai}\psi^\nu_j\right)\hat{\mathcal{D}}_\nu\phi_a{}^j{}_i \\
& + \frac{1}{192}e^{2\sigma}G_{\mu\nu\rho\sigma}\bar{\lambda}^{ai}\Upsilon^{\mu\nu\rho\sigma}\lambda_{ai} + \frac{i}{4\sqrt{2}}e^{-\sigma}F^I_{\mu\nu}\ell_I{}^j{}_i\bar{\lambda}^{ai}\Upsilon^{\mu\nu}\lambda_{aj} \\
& - \frac{i}{2}e^{-\sigma}\left(F^I_{\mu\nu}\ell_I{}^k{}_l\phi^{al}{}_k\phi_a{}^i{}_j + 2F^a_{\mu\nu}\phi_a{}^j{}_i\right)\Big[\frac{1}{4\sqrt{2}}\left(\bar{\psi}^i_\rho\Upsilon^{\mu\nu\rho\sigma}\psi_{\sigma j} + 2\bar{\psi}^{\mu i}\psi^\nu_j\right) \\
& + \frac{3}{20\sqrt{2}}\bar{\chi}^i\Upsilon^{\mu\nu}\chi_j - \frac{1}{4\sqrt{2}}\bar{\lambda}^{\alpha i}\Upsilon^{\mu\nu}\lambda_{\alpha j} + \frac{1}{2\sqrt{10}}\left(\bar{\chi}^i\Upsilon^{\mu\nu\rho}\psi_{\rho j} - 2\bar{\chi}^i\Upsilon^\mu\psi^\nu_j\right)\Big] \\
& + e^{-\sigma}F^a_{\mu\nu}\Big[\frac{1}{4}\left(2\bar{\lambda}^{ai}\Upsilon^\mu\psi^\nu_i - \bar{\lambda}^{ai}\Upsilon^{\mu\nu\rho}\psi_{\rho i}\right) + \frac{1}{2\sqrt{5}}\bar{\lambda}^{ai}\Upsilon^{\mu\nu}\chi_i\Big] \\
& + \frac{1}{4}e^{2\sigma}f_{bc}{}^a f_{dea}\phi^{bi}{}_k\phi^{ck}{}_j\phi^{dj}{}_l\phi^{el}{}_i - \frac{1}{2}e^\sigma f_{abc}\phi^{bi}{}_k\phi^{ck}{}_j\left(\bar{\psi}^j_\mu\Upsilon^\mu\lambda^a_i + \frac{2}{\sqrt{5}}\bar{\chi}^j\lambda_i{}^a\right) \\
& - \frac{i}{\sqrt{2}}e^\sigma f_{ab}{}^c\phi_c{}^i{}_j\bar{\lambda}^{aj}\lambda^b_i + \frac{i}{60\sqrt{2}}e^\sigma f_{ab}{}^c\phi^{al}{}_k\phi^{bj}{}_l\phi_c{}^k{}_j\Big(5\bar{\psi}^i_\mu\Upsilon^{\mu\nu}\psi_{\nu i} + 2\sqrt{5}\bar{\psi}^i_\mu\Upsilon^\mu\chi_i
\end{aligned}
$$

$$
+3\bar{\chi}^i\chi_i - 5\bar{\lambda}^{\alpha i}\lambda_{\alpha i}\Big) - \frac{1}{96}\epsilon^{\mu\nu\rho\sigma\kappa\lambda\tau}C_{\mu\nu\rho}F^a_{\sigma\kappa}F_{a\lambda\tau}\Bigg\} . \tag{C.32}
$$

The associated supersymmetry transformations have an expansion similar to that of the Lagrangian. Thus, the supersymmetry transformation of a field X takes the form

$$
\delta X = \delta^{(0)}X + h^2\delta^{(2)}X + h^4\delta^{(4)}X + \dots \ . \tag{C.33}
$$

We give the first two terms of this series for the gravity and $U(1)$ vector multiplet fields, and just the first term for the H vector multiplet fields. These terms are precisely those required to prove that the Lagrangian given in Eq. (C.32) is supersymmetric to order $h^2 \sim g_{\rm YM}^{-2}$. They are

$$
\begin{aligned}
\delta\sigma &= \frac{1}{\sqrt{5}}\bar{\chi}^i\varepsilon_i, \\
\delta e_\mu{}^{\underline{\nu}} &= \bar{\varepsilon}^i\Upsilon^{\underline{\nu}}\psi_{\mu i}, \\
\delta\psi_{\mu i} &= 2\hat{\mathcal{D}}_\mu\varepsilon_i - \frac{1}{80}\left(\Upsilon_\mu{}^{\nu\rho\sigma\eta} - \frac{8}{3}\delta^\nu_\mu\Upsilon^{\rho\sigma\eta}\right)\varepsilon_i G_{\nu\rho\sigma\eta}e^{2\sigma} + \frac{i}{5\sqrt{2}}\left(\Upsilon_\mu{}^{\nu\rho} - 8\delta^\nu_\mu\Upsilon^\rho\right)\varepsilon_j F^I_{\nu\rho}\ell_I{}^j{}_i e^{-\sigma} \\
&\quad + \frac{\kappa_7^2}{g_{\rm YM}^2}\left\{\frac{1}{2}\left(\phi_{ak}{}^j\hat{\mathcal{D}}_\mu\phi^a{}_i{}^k - \phi^a{}_i{}^k\hat{\mathcal{D}}_\mu\phi_{ak}{}^j\right)\varepsilon_j - \frac{i}{15\sqrt{2}}\Upsilon_\mu\varepsilon_i f_{ab}{}^c\phi^{al}{}_k\phi^{bj}{}_l\phi_c{}^k{}_j e^\sigma\right. \\
&\qquad \left. + \frac{i}{10\sqrt{2}}\left(\Upsilon_\mu{}^{\nu\rho} - 8\delta^\nu_\mu\Upsilon^\rho\right)\varepsilon_j\left(F^I_{\nu\rho}\ell_I{}^k{}_l\phi^{al}{}_k\phi_a{}^j{}_i + 2F^a_{\nu\rho}\phi_a{}^j{}_i\right)e^{-\sigma}\right\}, \\
\delta\chi_i &= \sqrt{5}\Upsilon^\mu\varepsilon_i\partial_\mu\sigma - \frac{1}{24\sqrt{5}}\Upsilon^{\mu\upsilon\rho\sigma}\varepsilon_i G_{\mu\nu\rho\sigma}e^{2\sigma} - \frac{i}{\sqrt{10}}\Upsilon^{\mu\nu}\varepsilon_j F^I_{\mu\nu}\ell_I{}^j{}_i e^{-\sigma} \\
&\quad + \frac{\kappa_7^2}{g_{\rm YM}^2}\left\{-\frac{i}{2\sqrt{10}}\Upsilon^{\mu\nu}\varepsilon_j\left(F^I_{\mu\nu}\ell_I{}^k{}_l\phi^{al}{}_k\phi_a{}^j{}_i + 2F^a_{\mu\nu}\phi_a{}^j{}_i\right)e^{-\sigma}\right. \\
&\quad \left. + \frac{i}{3\sqrt{10}}\varepsilon_i f_{ab}{}^c\phi^{al}{}_k\phi^{bj}{}_l\phi_c{}^k{}_j e^\sigma\right\}, \\
\delta C_{\mu\nu\rho} &= \left(-3\bar{\psi}^i_{[\mu}\Upsilon_{\nu\rho]}\varepsilon_i - \frac{2}{\sqrt{5}}\bar{\chi}^i\Upsilon_{\mu\nu\rho}\varepsilon_i\right)e^{-2\sigma}, \\
\ell_I{}^i{}_j\delta A^I_\mu &= \left[i\sqrt{2}\left(\bar{\psi}^i_\mu\varepsilon_j - \frac{1}{2}\delta^i_j\bar{\psi}^k_\mu\varepsilon_k\right) - \frac{2i}{\sqrt{10}}\left(\bar{\chi}^i\Upsilon_\mu\varepsilon_j - \frac{1}{2}\delta^i_j\bar{\chi}^k\Upsilon_\mu\varepsilon_k\right)\right]e^\sigma \\
&\quad + \frac{\kappa_7^2}{g_{\rm YM}^2}\left\{\left(\frac{i}{\sqrt{2}}\bar{\psi}^k_\mu\varepsilon_l - \frac{i}{\sqrt{10}}\bar{\chi}^k\Upsilon_\mu\varepsilon_l\right)\phi^{al}{}_k\phi_a{}^i{}_j e^\sigma - \bar{\varepsilon}^k\Upsilon_\mu\lambda^a_k\phi_a{}^i{}_j e^\sigma\right\}, \qquad\qquad \text{(C.34)} \\
\ell_I{}^\alpha\delta A^I_\mu &= \bar{\varepsilon}^i\Upsilon_\mu\lambda^\alpha_i e^\sigma, \\
\delta\ell_I{}^i{}_j &= -i\sqrt{2}\bar{\varepsilon}^i\lambda_{\alpha j}\ell_I{}^\alpha + \frac{i}{\sqrt{2}}\bar{\varepsilon}^k\lambda_{\alpha k}\ell_I{}^\alpha\delta^i_j \\
&\quad + \frac{\kappa_7^2}{g_{\rm YM}^2}\left\{\frac{i}{\sqrt{2}}\left[\bar{\varepsilon}^k\lambda_{\alpha l}\phi^{al}{}_k\phi_a{}^i{}_j\ell_I{}^\alpha + \bar{\varepsilon}^l\lambda_{ak}\phi^{ai}{}_j\ell_I{}^k{}_l - \left(\bar{\varepsilon}^i\lambda_{aj} - \frac{1}{2}\delta^i_j\bar{\varepsilon}^m\lambda_{am}\right)\phi^{al}{}_k\ell_I{}^k{}_l\right]\right\}, \\
\delta\ell_I{}^\alpha &= -i\sqrt{2}\bar{\varepsilon}^i\lambda^\alpha_j\ell_I{}^j{}_i + \frac{\kappa_7^2}{g_{\rm YM}^2}\left\{-\frac{i}{\sqrt{2}}\bar{\varepsilon}^i\lambda^\alpha_j\phi^{aj}{}_i\phi_a{}^l{}_k\ell_I{}^k{}_l\right\}, \\
\delta\lambda^\alpha_i &= -\frac{1}{2}\Upsilon^{\mu\nu}\varepsilon_i F^I_{\mu\nu}\ell_I{}^\alpha e^{-\sigma} + \sqrt{2}i\Upsilon^\mu\varepsilon_j p_\mu{}^{\alpha j}{}_i + \frac{\kappa_7^2}{g_{\rm YM}^2}\left\{\frac{i}{\sqrt{2}}\Upsilon^\mu\varepsilon_j\phi_{ai}{}^j p_\mu{}^{\alpha k}{}_l\phi^{al}{}_k\right\},
\end{aligned}
$$

$$
\begin{aligned}
\delta A^a_\mu &= \bar{\varepsilon}^i \Upsilon_\mu \lambda^a_i e^\sigma - \left(i\sqrt{2}\psi^i_\mu \varepsilon_j - \frac{2i}{\sqrt{10}} \bar{\chi}^i \Upsilon_\mu \varepsilon_j \right) \phi^{aj}{}_i e^\sigma , \\
\delta \phi_a{}^i{}_j &= -i\sqrt{2} \left(\bar{\varepsilon}^i \lambda_{aj} - \frac{1}{2} \delta^i_j \bar{\varepsilon}^k \lambda_{ak} \right) , \\
\delta \lambda^a_i &= \left(-\frac{1}{2} \Upsilon^{\mu\nu} \varepsilon_i \left(F^I_{\mu\nu} \ell_I{}^j{}_k \phi^{ak}{}_j + F^a_{\mu\nu} \right) e^{-\sigma} - i\sqrt{2} \Upsilon^\mu \varepsilon_j \hat{\mathcal{D}}_\mu \phi^a{}_i{}^j - i\varepsilon_j f^a{}_{bc} \phi^{bj}{}_k \phi^{ck}{}_i \right) , \\
\delta A^a_\mu &= \bar{\varepsilon}^i \Upsilon_\mu \lambda^a_i e^\sigma - \left(i\sqrt{2}\psi^i_\mu \varepsilon_j - \frac{2i}{\sqrt{10}} \bar{\chi}^i \Upsilon_\mu \varepsilon_j \right) \phi^{aj}{}_i e^\sigma , \\
\delta \phi_a{}^i{}_j &= -i\sqrt{2} \left(\bar{\varepsilon}^i \lambda_{aj} - \frac{1}{2} \delta^i_j \bar{\varepsilon}^k \lambda_{ak} \right) , \\
\delta \lambda^a_i &= -\frac{1}{2} \Upsilon^{\mu\nu} \varepsilon_i \left(F^I_{\mu\nu} \ell_I{}^j{}_k \phi^{ak}{}_j + F^a_{\mu\nu} \right) e^{-\sigma} - i\sqrt{2} \Upsilon^\mu \varepsilon_j \hat{\mathcal{D}}_\mu \phi^a{}_i{}^j - i\varepsilon_j f^a{}_{bc} \phi^{bj}{}_k \phi^{ck}{}_i .
\end{aligned}
$$

This completes our review of $\mathcal{N}=1$ EYM supergravity in seven dimensions.

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